IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Some properties of angular integrals

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2009 J. Phys. A: Math. Theor. 42 265201

(http://iopscience.iop.org/1751-8121/42/26/265201)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.154
The article was downloaded on 03/06/2010 at 07:54

Please note that terms and conditions apply.

J. Phys. A: Math. Theor. **42** (2009) 265201 (32pp) doi:10.1088/1751-8113/42/26/265201

Some properties of angular integrals

M Bergère and B Eynard

Institut de Physique Théorique, CEA/DSM/IPhT, CEA/Saclay, CNRS, URA 2306, F-91191 Gif-sur-Yvette Cedex, France

E-mail: michel.bergere@cea.fr and bertrand.eynard@cea.fr

Received 22 January 2009, in final form 24 April 2009
Published 5 June 2009
Online at stacks.iop.org/JPhysA/42/265201

Abstract
We find new representations for Itzykson–Zuber like angular integrals for arbitrary β, in particular for the orthogonal group $O(n)$, the unitary group $U(n)$ and the symplectic group $Sp(2n)$. We rewrite the Haar measure integral, as a flat Lebesgue measure integral, and we deduce some recursion formulae on n. The same methods give also Shatashvili's type moments. Finally, we prove that, in agreement with Brezin and Hikami's observation, the angular integrals are linear combinations of exponentials whose coefficients are polynomials in the reduced variables $(x_i - x_j)(y_i - y_j)$.

PACS number: 02.10.Yn
Mathematics Subject Classification: 15A52

1. Introduction

What we call an angular integral [29] is an integral over a compact Lie group $G_{\beta,n}$:

$$G_{1/2,n} = O(n), \qquad G_{1,n} = U(n), \qquad G_{2,n} = Sp(2n), \tag{1.1}$$

of the form

$$I_{\beta,n}(X,Y) = \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}}, \tag{1.2}$$

where X and Y are two given matrices, and $\mathrm{d}O$ is the Haar-invariant measure on the group. We shall also extend $I_{\beta,n}$ to arbitrary β (note that our β is half the one most commonly used in matrix models; for instance, we have $\beta = 1$ in the unitary case).

In this paper, we are going to consider the case where X and Y are diagonal matrices; however, let us first recall the Harish-Chandra case.

The Harish-Chandra case. In the case where X and Y are in the Lie algebra of the group [26] (i.e. real anti-symmetric in the $O(n)$ case, anti-Hermitian in the $U(n)$ case and quaternion-anti-self-dual in the $Sp(2n)$ case), the angular integral can be computed with the Weyl-character formula, and is given by the famous Harish-Chandra formula [14] (which is also a special case of the Duistermaat–Heckman localization [11]):

$$(X, Y) \in \text{Lie algebra} \Rightarrow \int \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} = C \sum_{w \in \mathrm{Weyl}} \frac{\mathrm{e}^{\mathrm{Tr}\, XY_w}}{\Delta_\beta(X)\Delta_\beta(Y_w)}, \tag{1.3}$$

where C is a normalization constant, w runs over the elements of the Weyl group, and the generalized Vandermonde determinant $\Delta_\beta(X)$ is the product of scalar products of positive roots with X (see [14, 26, 33] for details).

The diagonal case. However, for applications to many physics problems [13, 29], it would be more interesting to have X and Y in other representations, and in particular *X and Y diagonal matrices*.

Since an anti-Hermitian matrix is, up to a multiplication by i, a Hermitian matrix, and since every Hermitian matrix can be diagonalized with a unitary conjugation, for the unitary group, the Harish-Chandra formula applies as well to the case where X and Y are diagonal; this is known as the Itzykson–Zuber formula [22]:

$$\begin{cases} X = \mathrm{diag}(x_1, \ldots, x_n) \\ Y = \mathrm{diag}(y_1, \ldots, y_n) \end{cases} \Rightarrow \int_{U(n)} \mathrm{d}U\, \mathrm{e}^{\mathrm{Tr}\, XUYU^{-1}} = C_n \frac{\det \mathrm{e}^{x_i y_j}}{\Delta(X)\Delta(Y)}, \tag{1.4}$$

where $\Delta(X) = \Delta_1(X) = \prod_{i>j}(x_i - x_j)$ is the usual Vandermonde determinant.

For the other groups, computing angular integrals has remained an important challenge in mathematical physics for a rather long time. Much progress and many formulae have been found; however, a formula as compact and convenient as Harish-Chandra is still missing. And in particular, a formula which would allow us to compute multiple matrix integrals, generalizing that the method of Mehta [30], is still missing.

Calogero Hamiltonian. It is known that, in the diagonal case, $I_{\beta,n}$ satisfies the Calogero–Moser equation [2, 7], i.e. is an eigenfunction of the Calogero Hamiltonian:

$$H_{\mathrm{Calogero}}.I_{\beta,n} = \left(\sum_i y_i^2\right) I_{\beta,n}, \tag{1.5}$$

$$H_{\mathrm{Calogero}} = \sum_i \frac{\partial^2}{\partial x_i^2} + \beta \sum_{j \neq i} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \tag{1.6}$$

Many approaches towards computing angular integrals have used that differential equation. A basis of eigenfunctions of the Calogero Hamiltonian is the Hi–Jack polynomials [5, 7, 10, 28].

In particular, remarkable progress in the computation of $I_{\beta,n}$ was achieved recently by Brezin and Hikami [6]. By decomposing $I_{\beta,n}$ on the suitable basis of zonal polynomials, they were able to find a recursive algorithm to compute the terms in some power series expansion of $I_{\beta,n}$, and they obtained a remarkable structure. In particular, they observed that the power series reduces to a polynomial when $\beta \in \mathbb{N}$.

Morozov and Shatashvili's formulae. Another important question for physical applications is not only to compute the angular integral (the partition function in statistical physics language), but also all its moments, for instance,

$$M_{i,j} = \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} \|O_{i,j}\|^2 \tag{1.7}$$

and more generally for any indices $i_1, \ldots, i_{2k}, j_1, \ldots, j_{2k}$:

$$\int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} O_{i_1,i_2} O_{i_3,i_4} \ldots O_{i_{2k-1},i_{2k}} O^{-1}_{j_1,j_2} O^{-1}_{j_3,j_4} \ldots O^{-1}_{j_{2k-1},j_{2k}}. \tag{1.8}$$

In the $U(n)$ case $\beta = 1$, Morozov [32] found a beautiful formula for $M_{i,j}$, and Shatashvili [34] found a more general formula for any moments of type (1.8) using the action-angle variables of Gelfand–Tseytlin corresponding to the integrable structure of this integral.

For $\beta = 1/2, 1, 2$, in the Harish-Chandra case where X and Y are in the Lie algebra, a formula for all possible moments was also derived in [33], generalizing Morozov's one [15, 17].

In this paper, we shall propose new formulae for $M_{i,j}$ in the diagonal case for arbitrary β, and our method can also be generalized to all moments.

Outline of the paper

- Section 1 is an introduction, and we present a summary of the main results of this paper.
- In section 2 we set up the notations, and we review some known examples.
- In section 3, we show how to transform the angular integral with a Haar measure into a flat Lebesgue measure integral on a hyperplane. From it, we deduce a recursion formula, as well as a duality formula (the angular integral is an eigenfunction of kernel which is the Cauchy determinant to the power β).
- In section 4, we discuss the moments of the angular integral. We show that moments can also be obtained with Lebesgue measure integrals, and we show that they satisfy linear Dunkl-like equations. This can be used as a way to recover the Calogero equation for the angular integral.
- In section 5, we rewrite the angular integral as a symmetric sum of exponentials with polynomial prefactors. Those polynomials are called principal terms and can be computed recursively. In particular, we prove the conjecture of Brezin and Hikami [6] that the principal terms are polynomials in some reduced variables $(x_i - x_j)(y_i - y_j)$.
- In section 5.2, we prove a formula for $n = 3$ in terms of Bessel polynomials, and we propose a conjecture formula for arbitrary β and arbitrary n.
- In section 5.4, we focus on the symplectic case $\beta = 2$, for which we can improve the recursion formula.
- Section 6 is the conclusion.
- The appendices contain useful lemmas and proofs of the most technical theorems.

1.1. Summary of the main results presented in this paper

- We rewrite the angular integral with the Haar measure on the Lie group $G_{\beta,n}$, as a flat Lebesgue measure integral on its Lie algebra (notations are explained in section 3),

$$\boxed{I_{\beta,n}(X;Y) = \int \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} = \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^n \det(S - y_k X)^\beta},} \tag{1.9}$$

as well as its moments,

$$\boxed{\begin{aligned} M_{i,j} &= \int \mathrm{d}O \| O_{i,j} \|^2\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} \\ &= \beta \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^n \det(S - y_k X)^\beta} ((S - y_j X)^{-1})_{i,i}. \end{aligned}} \tag{1.10}$$

- We show that the $M_{i,j}$'s satisfy a linear functional equation (very similar to Dunkl operators):

$$\forall i, j, \quad \frac{\partial M_{i,j}}{\partial x_i} + \beta \sum_{k\neq i} \frac{M_{i,j} - M_{k,j}}{x_i - x_k} = M_{i,j} y_j \tag{1.11}$$

which implies the Calogero equation for $I_{\beta,n} = \sum_i M_{i,j} = \sum_j M_{i,j}$:

$$\sum_i \frac{\partial^2 I_{\beta,n}}{\partial x_i^2} + \beta \sum_{j\neq i} \frac{1}{x_i - x_j} \left(\frac{\partial I_{\beta,n}}{\partial x_i} - \frac{\partial I_{\beta,n}}{\partial x_j} \right) = \left(\sum_i y_i^2 \right) I_{\beta,n}. \tag{1.12}$$

Moreover, the integral of equation (1.10) is a solution of the linear functional equation (1.11) for any choice of integration domain (as long as there is no boundary term when one integrates by parts). We thus have a large set of solutions of the linear equation and also of the Calogero equation.

- We deduce a duality formula,

$$I_{\beta,n}(X; Y) = \det(X)^{1-\beta} \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \Delta(\Lambda)^{2\beta} \frac{I_{\beta,n}(X, \Lambda)}{\prod_{k=1}^n \prod_{j=1}^n (\lambda_j - y_k)^\beta}, \tag{1.13}$$

and a recursion formula,

$$\boxed{I_{\beta,n}(X; Y) = \frac{\mathrm{e}^{x_n \sum_{i=1}^n y_i}}{\prod_{i=1}^{n-1} (x_i - x_n)^{2\beta-1}} \int \mathrm{d}\lambda_1, \ldots \mathrm{d}\lambda_{n-1} \frac{I_{\beta,n-1}(X_{n-1}, \Lambda) \Delta(\Lambda)^{2\beta} \, \mathrm{e}^{-x_n \sum_i \lambda_i}}{\prod_{k=1}^n \prod_{i=1}^{n-1} (\lambda_i - y_k)^\beta},} \tag{1.14}$$

similar to that of [20].

- For $\beta \in \mathbb{N}$, the solution of the recursion can be written in terms of principal terms:

$$I_{\beta,n}(X, Y) = \sum_\sigma \frac{\mathrm{e}^{\sum_{i=1}^n x_i y_{\sigma(i)}}}{\Delta(X)^{2\beta} \Delta(Y_\sigma)^{2\beta}} \hat{\mathcal{I}}_{\beta,n}(X, Y_\sigma), \tag{1.15}$$

where $\sum_\sigma$ is the sum over all permutations.

The recursion relation (1.14) can be rewritten as a recursion for the principal terms $\hat{\mathcal{I}}_{\beta,n}(X, Y)$:

$$\boxed{\hat{\mathcal{I}}_{\beta,n}(X_n; Y_n) = \frac{\Delta(Y_n)^{2\beta}}{(\beta-1)!^{n-1}} \prod_{i=1}^{n-1} x_{i,n} \left(\frac{\partial}{\partial a_i} \right)^{\beta-1} \frac{\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1}, a) \, \mathrm{e}^{\sum_i x_{i,n}(a_i - y_i)}}{\prod_{k=1}^n \prod_{i=1,\neq k}^{n-1} (y_k - a_i)^\beta} \Bigg|_{a_i = y_i}.} \tag{1.16}$$

- For general n and β integers, we prove the conjecture of Brezin and Hikami [6] that the principal term $\hat{\mathcal{I}}_{\beta,n}(X, Y)$ is a symmetric polynomial of degree β in the $\tau_{i,j}$ variables:

$$\tau_{i,j} = -\frac{(x_i - x_j)(y_i - y_j)}{2}. \tag{1.17}$$

- In the case $n = 3$, we find this polynomial explicitly for any β (for β integer the sum is finite):

$$\boxed{I_{\beta,3} \propto \frac{\mathrm{e}^{x_1 y_1 + x_2 y_2 + x_3 y_3}}{(\Delta(x)\Delta(y))^\beta} \sum_{k=0}^\infty \frac{\Gamma(\beta - k)}{2^{6k} k! \Gamma(\beta + k)} \prod_{i<j} \mathcal{Y}_{\beta-1}^{(k)} \left(\frac{1}{\tau_{ij}} \right) + \text{sym},} \tag{1.18}$$

where $\mathcal{Y}_m$ is the mth Bessel polynomial, i.e. the modified Bessel function of the second kind (see the definition of $\mathcal{Y}_{\beta-1}$ in equation (2.4)).

- In the case $\beta = 2$ (i.e. symplectic group $Sp(2n)$), the recursion relation for the principal term can be written:

$$\begin{aligned}\hat{\mathcal{I}}_{2,n} &= \prod_{i=1}^{n-1}(x_i - x_n)(y_i - y_n)^2 \\ &\quad \times \prod_{i=1}^{n-1}\left(x_i - x_n - \sum_{k=1,\neq i}^{n}\frac{2}{y_i - y_k} + \frac{\partial}{\partial a_i}\right)\hat{\mathcal{I}}_{2,n-1}(X_{n-1}, a)|_{a_i=y_i} \\ &= \Delta(X_n)^2\Delta(Y_n)^2\frac{\det\left[X_{n-1} - x_n - \frac{2}{Y_{n-1}-y_n} + B + \partial_Y\right]}{\det(X_n - x_n)}\frac{\hat{I}_{2,n-1}(X_{n-1}; Y_{n-1})}{\Delta(X_{n-1})^2\Delta(Y_{n-1})^2},\end{aligned} \tag{1.19}$$

and B is the anti-symmetric matrix: $B_{i,j} = \frac{\sqrt{2}}{y_i - y_j}$, $B_{i,i} = 0$ and $\partial_Y = \operatorname{diag}\left(\partial_{y_1}, \ldots, \partial_{y_{n-1}}\right)$. In section 5.4, we propose an operator formalism to compute it, and we propose a conjecture formula in terms of decomposition into triangles.

2. Definitions and examples

2.1. Notations for angular integrals

Let X and Y be two diagonal matrices of size n:

$$X = \operatorname{diag}(x_1, \ldots, x_n), \qquad Y = \operatorname{diag}(y_1, \ldots, y_n). \tag{2.1}$$

We define the *angular integral*

$$I_{\beta,n}(x_1, \ldots, x_n; y_1, \ldots, y_n) = \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\operatorname{Tr} XOYO^{-1}}, \tag{2.2}$$

where $G_{\beta,n}$ denotes one of the Lie groups:

$$G_{1/2,n} = O(n), \qquad G_{1,n} = U(n), \qquad G_{2,n} = Sp(2n) \tag{2.3}$$

and $\mathrm{d}O$ is the invariant Haar measure on the corresponding compact Lie group.

We will later extend those notions to arbitrary values of β.

2.2. Bessel polynomials

For further use, we need to introduce some Bessel functions [1, 8, 27]. Those special functions are going to play a major role throughout this paper.

The Bessel polynomials (see [27]) $\mathcal{Y}_m(x)$ are defined by

$$\mathcal{Y}_m(x) = \sum_{k=0}^{\infty}\frac{\Gamma(m+k+1)}{k!\Gamma(m-k+1)}(x/2)^k = \sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{1/x}\mathcal{K}_{m+\frac{1}{2}}(1/x), \tag{2.4}$$

where $\mathcal{K}$ is the modified Bessel function of the second kind [1]. $\mathcal{Y}_m$ is a polynomial of degree m when m is an integer:

$$\mathcal{Y}_0 = 1, \qquad \mathcal{Y}_1 = x + 1, \qquad \mathcal{Y}_2 = 3x^2 + 3x + 1, \qquad \mathcal{Y}_3 = 15x^3 + 15x^2 + 6x + 1, \text{ etc.} \tag{2.5}$$

They satisfy

$$x^2\mathcal{Y}_m'' + (2x+2)\mathcal{Y}_m' - m(m+1)\mathcal{Y}_m = 0. \tag{2.6}$$

We shall also need

$$Q_{\beta,j}(x) = \sum_{k=0}^{\infty}\frac{\Gamma(\beta+j+k)}{k!\Gamma(\beta-j-k)}2^{-k}x^{\beta-j-k}, \tag{2.7}$$

which is a polynomial of degree $\beta - j$, if β is an integer.

In particular, $Q_{\beta,0}$ is the Carlitz polynomial [8] and is closely related to $\mathcal{Y}_{\beta-1}$:

$$Q_{\beta,0}(x) = x^\beta \mathcal{Y}_{\beta-1}\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^x x^{\beta+\frac{1}{2}} K_{\beta-\frac{1}{2}}(x), \tag{2.8}$$

satisfying

$$x^2 Q''_{\beta,0} - 2x(\beta + x) Q'_{\beta,0} + 2\beta(x+1) Q_{\beta,0} = 0. \tag{2.9}$$

The first few are

$$Q_{1,0} = x, \quad Q_{2,0} = x^2 + x, \quad Q_{3,0} = x^3 + 3x^2 + 3x, \quad Q_{4,0} = x^4 + 6x^3 + 15x^2 + 15x, \text{ etc.} \tag{2.10}$$

For higher j, the $Q_{\beta,j}$'s are the derivatives of Bessel polynomials:

$$Q_{\beta,j}(1/x) = 2^j x^{j-\beta} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathcal{Y}_{\beta-1}(x) = 2^j x^{j-\beta} \mathcal{Y}^{(j)}_{\beta-1}(x). \tag{2.11}$$

They satisfy

$$-x Q_{\beta,j} = \frac{1}{4} Q_{\beta,j+1} + j Q_{\beta,j} + (j-\beta)(j+\beta-1) Q_{\beta,j-1}, \tag{2.12}$$

$$Q_{\beta,j+1} = 2\left(\beta - j - x\frac{\mathrm{d}}{\mathrm{d}x}\right) Q_{\beta,j}. \tag{2.13}$$

The first few are

$$Q_{2,1} = 2x, \qquad Q_{3,1} = 6x^2 + 12x, \qquad Q_{4,1} = 12x^3 + 60x^2 + 90x, \tag{2.14}$$

$$Q_{3,2} = 24x, \qquad Q_{4,2} = 120x^2 + 360x, \qquad Q_{4,3} = 720x, \quad \text{etc.} \tag{2.15}$$

2.3. *Examples of angular integrals with $n = 1, 2, 3$*

- $n = 1$. The $n = 1$ case needs no computation, and gives

$$I_{\beta,1}(x; y) = \mathrm{e}^{xy}. \tag{2.16}$$

- $n = 2$. The $n = 2$ case requires a little bit of easy computation, and it has been known for some time that we have (this formula is rederived in this paper):

$$\begin{aligned} I_{\beta,2}(X, Y) &= \frac{\mathrm{e}^{x_1y_1+x_2y_2}}{\tau^\beta} \mathcal{Y}_{\beta-1}(1/\tau) + \frac{\mathrm{e}^{x_1y_2+x_2y_1}}{(-\tau)^\beta} \mathcal{Y}_{\beta-1}(-1/\tau) \\ &= \frac{\mathrm{e}^{x_1y_1+x_2y_2}}{\tau^{2\beta}} Q_{\beta,0}(\tau) + \frac{\mathrm{e}^{x_1y_2+x_2y_1}}{(-\tau)^{2\beta}} Q_{\beta,0}(-\tau), \end{aligned} \tag{2.17}$$

where

$$\tau = -\tfrac{1}{2}(x_1 - x_2)(y_1 - y_2). \tag{2.18}$$

It can also be written in terms of the modified Bessel function $\mathcal{I}$:

$$I_{\beta,2}(X; Y) = \frac{\mathrm{e}^{\frac{1}{2}(x_1+x_2)(y_1+y_2)}}{\tau^{2\beta-1}} \mathcal{I}_{\beta-\frac{1}{2}}(\tau), \tag{2.19}$$

where

$$\mathcal{I}_m(\tau) = (\tau/2)^{2m} \sum_{k=0}^{\infty} \frac{(\tau/2)^{2k}}{k!\,\Gamma(m+k+1)}, \qquad \mathcal{I}_m = \mathcal{I}''_m + \frac{1-2m}{\tau} \mathcal{I}'_m. \tag{2.20}$$

- $n = 3$. We show in this paper that (proof in appendix C)

$$\boxed{I_{\beta,3} \propto \frac{e^{x_1y_1+x_2y_2+x_3y_3}}{(\Delta(x)\Delta(y))^\beta} \sum_{k=0}^{\infty} \frac{\Gamma(\beta-k)}{2^{6k}k!\Gamma(\beta+k)} \prod_{i<j} \mathcal{Y}^{(k)}_{\beta-1}\left(\frac{1}{\tau_{ij}}\right) + \text{perm.},} \tag{2.21}$$

where

$$\tau_{i,j} = -\frac{(x_i - x_j)(y_i - y_j)}{2} \tag{2.22}$$

and +perm. means that we have to symmetrize over all permutations of the y_j's. Note that the sum is finite and stops at $k = \beta - 1$ if β is an integer.
This formula was already obtained by [4] for $\beta = 1/2$.

- $n > 3$. We show in this paper that for arbitrary n and β, the angular integral is of the form conjectured by Brezin and Hikami

$$I_{\beta,n} \propto \frac{e^{\sum_i x_i y_i}}{(\Delta(x)\Delta(y))^{2\beta}} \hat{\mathcal{I}}_{\beta,n}(\tau_{ij}) + \text{perm.}, \tag{2.23}$$

where $\hat{\mathcal{I}}_{\beta,n}(\tau_{ij})$ is a polynomial in the $\tau_{i,j}$'s, and for which we write a recursion relation.

3. Transformation of the angular integral

In this section, we transform the Haar measure group integral into a flat Lebesgue measure integral. We use the same idea as in [19] for $\beta = 1/2$, but for the arbitrary half-integer β.

3.1. Lagrange multipliers

For $\beta = 1/2, 1, 2$, an element, $O \in G_{\beta,n}$, is an orthonormal basis, i.e. a collection of n orthonormal vectors $e_1, \ldots, e_n$, whose coordinates, $O_{i,j} = (e_i)_j$, are of the form

$$(e_i)_j = O_{i,j} = \sum_{\alpha=0}^{2\beta-1} (e_i)^\alpha_j \epsilon_\alpha, \tag{3.1}$$

where the ϵ_α's form a basis of a Clifford algebra (indeed, this reproduces the three groups $G_{\beta,n}$ for $\beta = 1/2, 1, 2$),

$$\epsilon_0 = 1, \epsilon_0^\dagger = \epsilon_0, \qquad \forall\alpha > 0: \quad \epsilon_\alpha^2 = -1, \epsilon_\alpha^\dagger = -\epsilon_\alpha, \qquad \epsilon_\alpha.\epsilon_{\alpha'} = -\epsilon_{\alpha'}.\epsilon_\alpha, \tag{3.2}$$

with structure constants (only for $\beta = 2$)

$$\epsilon_\alpha \epsilon^\dagger_{\alpha'} = \sum_{\alpha''} \eta_{\alpha,\alpha',\alpha''} \epsilon_{\alpha''}, \tag{3.3}$$

where $\eta_{\alpha,\alpha',\alpha''}$ has the property useful for our purpose that for every pair (α, α'), there is exactly only one α'' such that $\eta_{\alpha,\alpha',\alpha''} \neq 0$. In particular, $\eta_{\alpha,\alpha,\alpha''} = \delta_{\alpha'',0}$.

The basis must be orthonormal, i.e.,

$$e_i.e_j^\dagger = \delta_{i,j} = \sum_{k=1}^{n} (e_i)_k (e_j)^\dagger_k = \sum_{k=1}^{n} \sum_{\alpha,\alpha'=0}^{2\beta-1} (e_i)^\alpha_k (e_j)^{\alpha'}_k \epsilon_\alpha \epsilon^\dagger_{\alpha'}. \tag{3.4}$$

We introduce Lagrange multipliers to enforce those orthonormality relations:

$$\delta\big(e_i.e_i^\dagger - 1\big) = \int dS_{i,i}\, e^{S_{i,i}(1-\sum_{k,\alpha}((e_i)^\alpha_k)^2} \tag{3.5}$$

and if $i < j$

$$\delta\left(e_i.e_j^\dagger\right) = \int\ldots\int \mathrm{d}S_{i,j}^0,\ldots,\mathrm{d}S_{i,j}^{2\beta-1}\,\mathrm{e}^{-2\sum_{\alpha,\alpha',\alpha''}\sum_k S_{i,j}^\alpha((e_i)_k^{\alpha'})((e_j)_k^{\alpha''})\eta_{\alpha',\alpha'',\alpha}}$$
$$= \int \mathrm{d}S_{i,j}\,\mathrm{e}^{-2\sum_k S_{i,j}((e_i)_k)((e_j)_k^\dagger)}, \tag{3.6}$$

where each integral is over the imaginary axis.

Since the scalar product is invariant under group transformations (i.e. change of orthogonal basis), the following measure is invariant and thus must be proportional to the Haar measure:

$$\mathrm{d}O \propto \prod_{i,j,\alpha}\mathrm{d}(e_i)_j^\alpha \prod_i \delta\left(e_i.e_i^\dagger - 1\right)\prod_{i<j}\delta\left(e_i.e_j^\dagger\right), \tag{3.7}$$

i.e.

$$\mathrm{d}O \propto \prod_{i,j,\alpha}\mathrm{d}(e_i)_j^\alpha \int \mathrm{d}S\,\mathrm{e}^{\sum_i S_{i,i}}\,\mathrm{e}^{-\sum_i\sum_k S_{i,i}|(e_i)_k|^2}\,\mathrm{e}^{-2\sum_{i<j}\sum_k S_{i,j}^\alpha(e_i)_k(e_j)_k^\dagger}, \tag{3.8}$$

where

$$\mathrm{d}S = \prod_i \mathrm{d}S_{i,i}\prod_{i<j}\mathrm{d}S_{i,j} = \prod_{i=1}^n \mathrm{d}S_{i,i}\prod_{i<j}\prod_{\alpha=0}^{2\beta-1}\mathrm{d}S_{i,j}^\alpha \tag{3.9}$$

is the $G_{\beta,n}$ invariant measure on the space $E_{\beta,n}$:

$$\mathrm{i}S \in \begin{cases} E_{1/2,n} = \{n\times n \text{ real symmetric matrices}\} \\ E_{1,n} = \{n\times n \text{ Hermitian matrices}\} \\ E_{2,n} = \{n\times n \text{ quaternion self} - \text{dual matrices}\}, \end{cases} \tag{3.10}$$

where we have completed S by self-duality ($S = S^\dagger$):

$$S_{j,i}^0 = S_{i,j}^0 \qquad \text{and} \qquad \forall\alpha > 0 \quad S_{j,i}^\alpha = -S_{i,j}^\alpha. \tag{3.11}$$

Therefore, we have (up to a multiplicative constant)

$$I_{\beta,n}(X;Y) \propto \int \mathrm{d}S\int \mathrm{d}e_1\ldots\mathrm{d}e_n\,\mathrm{e}^{\sum_i S_{i,i}}\,\mathrm{e}^{\sum_{i,k}x_i y_k|(e_i)_k|^2}$$
$$\times\,\mathrm{e}^{-\sum_i\sum_k S_{i,i}|(e_i)_k|^2}\,\mathrm{e}^{-2\sum_{i<j}\sum_k\sum_{\alpha,\alpha',\alpha''}S_{i,j}^\alpha((e_i)_k^{\alpha'})((e_j)^{\dagger\alpha''}_k)\eta_{\alpha',\alpha'',\alpha}}. \tag{3.12}$$

The integral over the $(e_i)_k^\alpha$'s is now Gaussian and can be performed. The Gaussian integrals for each k are independent.

The quadratic form in the exponential is, for each k,

$$\sum_{\alpha,\alpha',\alpha''}\sum_{i,j}(e_i)_k^\alpha(e_j)_k^{\alpha'}\eta_{\alpha,\alpha',\alpha''}\left(\delta_{i,j}x_i y_k\delta_{\alpha'',0} - S_{i,j}^{\alpha''}\right). \tag{3.13}$$

If we define the vector $v_k = (v_{1,k},\ldots,v_{n,k})$ where $v_{i,k} = \sum_\alpha (e_i)_k^\alpha \epsilon_\alpha^\dagger$, we have to compute the Gaussian integral

$$\int \mathrm{d}v_k\,\mathrm{e}^{-v_k^\dagger(S-y_kX)v_k}. \tag{3.14}$$

For the three values of $\beta = 1/2, 1, 2$, this integral is worth:

$$\int \mathrm{d}v_k\,\mathrm{e}^{-v_k^\dagger(S-y_kX)v_k} = \frac{(2\pi)^\beta}{\det(S-y_kX)^\beta}, \tag{3.15}$$

where det is the product of singular values (see [29]).

Thus we get the following theorem.

Theorem 3.1. *The angular integral $I_{\beta,n}(X;Y)$ is also equal to the following flat Lebesgue measure integral:*

$$\boxed{I_{\beta,n}(X;Y) \propto \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^{n} \det(S - y_k X)^{\beta}} = \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_{k=1}^{n} \det(S - y_k)^{\beta}}.} \tag{3.16}$$

In the last formula, we have made the change of variable $S \to X^{1/2} S X^{1/2}$. Also, the integration domain for S, which was $\mathrm{i}E_{\beta,n}$ before exchanging the integrations over S and e, is now shifted to the right, so that all singular values of $(S - y_k X)$ have positive real part. The integration domain for S can be deformed such that the integral remains convergent and the integration path goes to the right of all zeroes of the denominator. If β is a half-integer or an integer, the denominator is not singular near ∞, and the integration contour can be closed. This will be made more precise below.

Remark 1. For the moment, this formula holds only for $\beta = 1/2, 1, 2$. Later we will extend it to other values of β.

Remark 2. Another remark is that a similar formula can be obtained by exchanging the roles of X and Y.

3.2. The duality formula

Note that the matrix S itself can be diagonalized with a $G_{\beta,n}$ conjugation:

$$S = O\Lambda O^{-1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n),\ O \in G_{\beta,n}, \tag{3.17}$$

and the measure $\mathrm{d}S$ is up to a constant [29]:

$$\mathrm{d}S \propto \mathrm{d}O \mathrm{d}\Lambda \Delta(\Lambda)^{2\beta}. \tag{3.18}$$

Therefore, the angular integral reappears in the RHS:

$$\begin{aligned} I_{\beta,n}(X;Y) &\propto \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_{k=1}^{n} \det(S - y_k)^{\beta}} \\ &\propto \det(X)^{1-\beta} \int \mathrm{d}\Lambda \Delta(\Lambda)^{2\beta} \frac{I_{\beta,n}(X,\Lambda)}{\prod_{k=1}^{n} \prod_{j=1}^{n} (\lambda_j - y_k)^{\beta}}. \end{aligned} \tag{3.19}$$

Here, if we assume that $\forall i,\ x_i \in \mathbb{R}^{+}$, the integration contours for the λ_i's are of the form $r + \mathrm{i}\mathbb{R}$ where $r > \max(\mathrm{Re} y_k)$. If β is an integer or a half-integer, the denominator in the integrand is not singular near ∞, and the integration contour can be closed. Thus, if 2β is an integer, the integration contours for the λ_i's can be chosen as circles of radius $> \max(|y_k|)$.

This equation looks better if we rewrite it in terms of the Cauchy determinant $D_n(X, Y)$,

$$D_n(X,Y) = \det\left(\frac{1}{x_i - y_j}\right) = \frac{\Delta(X)\Delta(Y)}{\prod_{i,j}(x_i - y_j)}, \tag{3.20}$$

and the rescaled function,

$$\check{I}_{\beta,n}(x_1,\dots,x_n; y_1,\dots,y_n) = (\Delta(X)\Delta(Y))^{\beta} I_{\beta,n}(x_1,\dots,x_n; y_1,\dots,y_n). \tag{3.21}$$

We then have

Theorem 3.2. *The rescaled function $\check{I}_{\beta,n}$ satisfies the duality formula,*

$$\boxed{\check{I}_{\beta,n}(X;Y) \propto \det(X)^{1-\beta} \int \mathrm{d}\Lambda\, \check{I}_{\beta,n}(X,\Lambda) D_n(\Lambda, Y)^{\beta},} \tag{3.22}$$

i.e. $\check{I}_{\beta,n}(X;Y)$ is an eigenfunction of the kernel D_n^{β}.

Remark 3. The duality formula above was derived for $\beta = 1/2, 1, 2$, but it makes sense for any β.

Remark 4. It is easy to check that this relation is satisfied for the Itzykson–Zuber case $\beta = 1$; indeed, in that case we have $\check{I}_{1,n}(X;Y) = \det(\mathrm{e}^{x_i y_j}) = \sum_\rho (-1)^\rho \prod_i \mathrm{e}^{y_i x_{\rho(i)}}$, and

$$\begin{aligned}\int \mathrm{d}\Lambda \check{I}_{1,n}(X,\Lambda) D_n(\Lambda, Y) &\propto \sum_{\sigma,\rho} (-1)^\sigma (-1)^\rho \int \prod_{i=1}^n \frac{\mathrm{e}^{\lambda_i x_{\rho(i)}}}{\lambda_i - y_{\sigma(i)}} \mathrm{d}\lambda_i \\ &= \sum_{\sigma,\rho} (-1)^\sigma (-1)^\rho \prod_{i=1}^n \mathrm{e}^{y_{\sigma(i)} x_{\rho(i)}} \\ &= n!\det(\mathrm{e}^{x_i y_j}) \\ &\propto \check{I}_{1,n}(X,Y). \end{aligned} \tag{3.23}$$

3.3. The recursion formula

First, let us note that we can always assume that $x_n = 0$, otherwise we perform a shift, $X \to X - x_n$:

$$\begin{aligned} I_{\beta,n}(X,Y) &= \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} \\ &= \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}(X-x_n)OYO^{-1}}\, \mathrm{e}^{x_n \,\mathrm{Tr}\, OYO^{-1}} \\ &= \mathrm{e}^{x_n \,\mathrm{Tr}\, Y} \int_{G_{\beta,n}} \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}(X-x_n)OYO^{-1}}. \end{aligned} \tag{3.24}$$

Thus we define

$$X_{n-1} = \mathrm{diag}(x_1, \ldots, x_{n-1}), \qquad \tilde{X} = X_{n-1} - x_n \mathrm{Id}_{n-1}. \tag{3.25}$$

Then, we note that the orthonormality of the basis e_i,

$$e_i . e_j^\dagger = \delta_{i,j}, \tag{3.26}$$

implies that if we already know $e_1, \ldots, e_{n-1}$, then e_n is completely fixed (up to an irrelevant phase). In other words, it is sufficient to enforce only the orthonormality of $e_1, \ldots, e_{n-1}$ with Lagrange multipliers, i.e. introduce a matrix S of size $n-1$.

Also, because of our shift $X \to X - x_n$, we note that e_n does not appear in the integrand.

Then, we write as in equation (3.8)

$$\begin{aligned} \mathrm{d}O &\propto \prod_{i=1}^{n-1} \mathrm{d}e_i \prod_{i=1}^{n-1} \delta(e_i.e_i - 1) \prod_{i<j=1}^{n-1} \delta\big(e_i.e_j^\dagger\big) \\ &\propto \prod_{i=1}^{n-1}\prod_{j=1}^{n}\prod_{\alpha=0}^{2\beta-1} \mathrm{d}(e_i)_j^\alpha \int_{iE_{\beta,n-1}} \mathrm{d}S\, \mathrm{e}^{\sum_i S_{i,i}}\, \mathrm{e}^{-\sum_i\sum_k S_{i,i}|(e_i)_k|^2}\, \mathrm{e}^{-2\sum_{i<j}\sum_k S_{i,j}(e_i)_k (e_j)_k^\dagger} \\ &\propto \prod_{i=1}^{n-1} \mathrm{d}e_i \int_{iE_{\beta,n-1}} \mathrm{d}S\, \mathrm{e}^{\mathrm{Tr}\, S}\, \mathrm{e}^{-\sum_{i,j} S_{i,j} e_i . e_j^\dagger}, \end{aligned} \tag{3.27}$$

which implies, after performing the Gaussian integral over the $e_1, \ldots, e_{n-1}$,

$$\begin{aligned} I_{\beta,n}(X;Y) &\propto \mathrm{e}^{x_n \,\mathrm{tr}\, Y} \int_{iE_{\beta,n-1}} \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^n \det(S - y_k\tilde{X})^\beta} \\ &\propto \frac{\mathrm{e}^{x_n \,\mathrm{tr}\, Y}}{\prod_{i=1}^{n-1}(x_i - x_n)^{2\beta-1}} \int_{iE_{\beta,n-1}} \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S\tilde{X}}}{\prod_{k=1}^n \det(S - y_k)^\beta}. \end{aligned} \tag{3.28}$$

Again, S can be diagonalized:

$$S = O\Lambda O^{-1}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}), \qquad O \in G_{\beta,n-1}, \tag{3.29}$$

i.e. the rank n angular integral $I_{\beta,n}$ is expressed in terms of the rank $n-1$:

$$\begin{aligned} I_{\beta,n}(X;Y) &\propto \frac{\mathrm{e}^{x_n \operatorname{tr} Y}}{\prod_{i=1}^{n-1}(x_i - x_n)^{2\beta-1}} \int \mathrm{d}\Lambda \frac{I_{\beta,n-1}(\tilde{X}, \Lambda)\Delta(\Lambda)^{2\beta}}{\prod_{k=1}^{n}\prod_{i=1}^{n-1}(\lambda_i - y_k)^{\beta}} \\ &\propto \frac{\mathrm{e}^{x_n \operatorname{tr} Y}}{\prod_{i=1}^{n-1}(x_i - x_n)^{2\beta-1}} \int \mathrm{d}\Lambda \frac{I_{\beta,n-1}(X_{n-1}, \Lambda)\Delta(\Lambda)^{2\beta}\,\mathrm{e}^{-x_n\sum_i \lambda_i}}{\prod_{k=1}^{n}\prod_{i=1}^{n-1}(\lambda_i - y_k)^{\beta}}, \end{aligned} \tag{3.30}$$

which is our main recursion formula.

Theorem 3.3. *The angular integrals $I_{\beta,n}(X;Y)$ satisfy the recursion*

$$\boxed{I_{\beta,n}(X;Y) \propto \frac{\mathrm{e}^{x_n \sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n-1}(x_i - x_n)^{2\beta-1}} \int \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_{n-1} \frac{I_{\beta,n-1}(X_{n-1}, \Lambda)\Delta(\Lambda)^{2\beta}\,\mathrm{e}^{-x_n\sum_i \lambda_i}}{\prod_{k=1}^{n}\prod_{i=1}^{n-1}(\lambda_i - y_k)^{\beta}}.} \tag{3.31}$$

Here again, the integration contours for the λ_i's are such that the integral is convergent and such that they surround all the y_k's. For instance, if 2β is an integer, and if $\forall i\, x_i \in \mathbb{R}^+$, the integration contour for the λ_i's can be chosen as circles of radius $> \max(|y_k|)$.

Remark 5. Now this recursion formula can be used to define $I_{\beta,n}$ for arbitrary β, so that it coincides with the angular integral for $\beta = 1/2, 1, 2$.

We have also the iterated form,

$$\begin{aligned} I_{\beta,n}(X;Y) \propto{}& \frac{\mathrm{e}^{x_n \sum_{i=1}^{n} y_i}}{\Delta(X)^{2\beta-1}} \int \prod_{i=1}^{n-1}\prod_{j=1}^{i} \mathrm{d}\lambda_{i,j} \\ &\times \frac{\prod_{i=1}^{n-1}\prod_{1\leqslant j<j'\leqslant i}(\lambda_{i,j'} - \lambda_{i,j})^{2\beta}\prod_{i=1}^{n-1}\mathrm{e}^{(x_i - x_{i+1})\sum_j \lambda_{i,j}}}{\prod_{i=1}^{n-1}\prod_{j=1}^{i+1}\prod_{j'=1}^{i}(\lambda_{i,j'} - \lambda_{i+1,j})^{\beta}}, \end{aligned} \tag{3.32}$$

where we have defined $\lambda_{n,j} = y_j$, and where the integration contours are circles such that

$$|\lambda_{i,j}| = \rho_i, \qquad \rho_1 > \rho_2 > \cdots > \rho_{n-1} > \max|y_j|. \tag{3.33}$$

Remark 6. A similar recursion relation was also found in [19–21], but the authors found real integrals instead of contour integrals. The advantage of our formulation is that we can easily move integration contours and find new relations, as we will see below.

4. Moments of angular integrals and Calogero

In this section, we compute moments of the angular integral, and we show that our formula indeed satisfies the Calogero equation.

4.1. The generalized Morozov's formula

Define the quadratic moments (see [32] for $\beta = 1$):

$$M_{i,j} = \int_{G_{\beta,n}} \mathrm{d}O \, \| O_{i,j} \|^2 \, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}}. \tag{4.1}$$

The same calculation as above yields

$$\begin{aligned} M_{i,j} &= \beta \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^n \det(S - y_k X)^\beta} ((S - y_j X)^{-1})_{i,i} \\ &= \frac{\beta}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_{k=1}^n \det(S - y_k)^\beta} ((S - y_j)^{-1})_{i,i}. \end{aligned} \tag{4.2}$$

As a consistency check and as a warmup exercise, let us show that this formula satisfies

$$I_{\beta,n} = \sum_j M_{i,j}, \tag{4.3}$$

which comes from $\forall O \in G_{\beta,n}$, $\sum_j \| O_{i,j} \|^2 = 1$.

We have

$$\begin{aligned} \sum_j M_{i,j} &= \sum_j \frac{\beta}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_{k=1}^n \det(S - y_k)^\beta} ((S - y_j)^{-1})_{i,i} \\ &= \frac{-1}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \, \mathrm{e}^{\mathrm{Tr}\, SX} \frac{\partial}{\partial S_{i,i}} \frac{1}{\prod_{k=1}^n \det(S - y_k)^\beta} \\ &= \frac{1}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{1}{\prod_{k=1}^n \det(S - y_k)^\beta} \frac{\partial}{\partial S_{i,i}} \mathrm{e}^{\mathrm{Tr}\, SX} \\ &= \det(X)^{1-\beta} \int \mathrm{d}S \frac{1}{\prod_{k=1}^n \det(S - y_k)^\beta} \mathrm{e}^{\mathrm{Tr}\, SX} \\ &= I_{\beta,n}. \end{aligned} \tag{4.4}$$

Note that this equality holds independently of the integration domain of S, provided that one can integrate by parts without picking boundary terms.

Remark 7. Of course a similar equation can be found by exchanging the roles of X and Y, and one gets symmetrically

$$\sum_i M_{i,j} = I_{\beta,n}. \tag{4.5}$$

4.2. Other moments

Since, after introducing the Lagrange multipliers, the integral becomes Gaussian in the $O_{i,j}$'s, any polynomial moment can be computed using Wick's theorem. It is sufficient to compute the propagator

$$\left\langle O_{i,k} O_{j,l}^\dagger \right\rangle = \beta \delta_{k,l} ((S - y_k X)^{-1})_{i,j}. \tag{4.6}$$

Then, the expectation value of any polynomial moment is obtained as the sum over all pairings of the product of propagators.

For instance,

$$\int \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, XOYO^{-1}} O_{i_1,j_1} O_{i_2,j_2} O^\dagger_{i_3,j_3} O^\dagger_{i_4,j_4}$$

$$= \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, S}}{\prod_{k=1}^n \det(S - y_k X)^\beta} \big[\delta_{j_1,j_3}\delta_{j_2,j_4}((S - y_{j_1}X)^{-1})_{i_1,i_3}((S - y_{j_2}X)^{-1})_{i_2,i_4}$$
$$+ \delta_{j_1,j_4}\delta_{j_2,j_3}((S - y_{j_1}X)^{-1})_{i_1,i_4}((S - y_{j_2}X)^{-1})_{i_2,i_3}\big]. \tag{4.7}$$

In principle, one could compute with this method the generalization of all Shatashvili's moments [34].

4.3. Linear equations

In this section, we prove that the $M_{i,j}$'s satisfy the following linear functional relations, which are very similar to Dunkl equations [11]:

$$\boxed{\forall\, i,\, j, \qquad \frac{\partial M_{i,j}}{\partial y_j} + \beta \sum_{l\neq j} \frac{M_{i,l} - M_{i,j}}{y_l - y_j} = M_{i,j} x_i.} \tag{4.8}$$

We are going to give two different proofs of equation (4.8). The first one below is based on integration by parts. It can be done for the three groups $\beta = 1/2, 1, 2$; however, it is rather tedious for $\beta = 1/2$ and $\beta = 2$, and we present the proof only for $\beta = 1$. Another proof valid for all three values of β is presented in section 4.6.

Let us check that equation (4.2) satisfies equation (4.8) (for $\beta = 1$). We first rewrite

$$\frac{1}{y_l - y_j}\left(\frac{1}{S - y_l} - \frac{1}{S - y_j}\right)_{i,i} = ((S - y_l)^{-1}(S - y_j)^{-1})_{i,i}$$
$$= \sum_m ((S - y_l)^{-1})_{i,m}((S - y_j)^{-1})_{m,i}. \tag{4.9}$$

For $\beta = 1$, we may consider all variables $S_{i,m}$ to be independent variables, and we integrate by parts:

$$\sum_{l\neq j} \frac{M_{i,l} - M_{i,j}}{y_l - y_j}$$
$$= \sum_{l\neq j}\sum_m \frac{\beta}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_{k=1}^n \det(S - y_k)^\beta}((S - y_l)^{-1})_{i,m}((S - y_j)^{-1})_{m,i}$$
$$= -\sum_m \frac{1}{x_i}\det(X)^{1-\beta}\int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\det(S - y_j)^\beta}((S - y_j)^{-1})_{m,i}\frac{\partial}{\partial S_{i,m}}\frac{1}{\prod_{l\neq j}\det(S - y_l)^\beta}$$
$$= \sum_m \frac{1}{x_i}\det(X)^{1-\beta}\int \mathrm{d}S \frac{1}{\prod_{l\neq j}\det(S - y_l)^\beta}\frac{\partial}{\partial S_{i,m}}\frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\det(S - y_j)^\beta}((S - y_j)^{-1})_{m,i}$$
$$= \sum_m \frac{1}{x_i}\det(X)^{1-\beta}\int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S - y_k)^\beta}((S - y_j)^{-1})_{m,i}x_i\delta_{i,m}$$
$$- \beta\sum_m \frac{1}{x_i}\det(X)^{1-\beta}\int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S - y_k)^\beta}((S - y_j)^{-1})_{i,m}((S - y_j)^{-1})_{m,i}$$
$$- \sum_m \frac{1}{x_i}\det(X)^{1-\beta}\int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S - y_k)^\beta}((S - y_j)^{-1})_{m,m}((S - y_j)^{-1})_{i,i}$$

$$
\begin{aligned}
&= \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S-y_k)^\beta} ((S-y_j)^{-1})_{i,i} \\
&\quad - \frac{1}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S-y_k)^\beta} ((S-y_j)^{-2})_{i,i} \\
&\quad - \frac{\beta}{x_i} \det(X)^{1-\beta} \int \mathrm{d}S \frac{\mathrm{e}^{\mathrm{Tr}\, SX}}{\prod_k \det(S-y_k)^\beta} \mathrm{Tr}\,(S-y_j)^{-1}((S-y_j)^{-1})_{i,i} \\
&= \frac{1}{\beta}\left(x_i M_{i,j} - \frac{\partial M_{i,j}}{\partial y_j}\right) \qquad \text{QED.}
\end{aligned} \tag{4.10}
$$

The same computation can be repeated for $\beta = 1/2$ and $\beta = 2$, with additional steps because the variables $S_{i,m}$ are no longer independent, and also because for $\beta = 2$, $\det(S - y_j)$ is defined as the product of singular values. Another proof is given in section 4.6.

Remark 8. Of course, a similar equation can be found by exchanging the roles of X and Y, and one gets the symmetric linear equation:

$$\forall i, j, \qquad \frac{\partial M_{i,j}}{\partial x_i} + \beta \sum_{l \neq i} \frac{M_{l,j} - M_{i,j}}{x_l - x_i} = M_{i,j} y_j. \tag{4.11}$$

Remark 9. Again, this proves that equation (4.2) is the solution of the differential equation (4.8), for any choice of integration domain provided that we can integrate by parts. In fact, by taking linear combinations of all possible integration contours, we get the general solution of the linear equation (4.8). However, a general solution of equation (4.8) is not necessarily symmetric in X and Y and does not necessarily obey equation (4.11).

4.4. The Calogero equation

Here, we prove that $I_{\beta,n}$ satisfies the Calogero equation.

Start from the linear equation,

$$\frac{\partial M_{i,j}}{\partial x_i} + \beta \sum_{k \neq i} \frac{M_{i,j} - M_{k,j}}{x_i - x_k} = M_{i,j} y_j, \tag{4.12}$$

then sum over j using equation (4.3):

$$\frac{\partial I}{\partial x_i} = \sum_j M_{i,j} y_j. \tag{4.13}$$

Then apply $\frac{\partial}{\partial x_i}$:

$$
\begin{aligned}
\frac{\partial^2 I}{\partial x_i^2} &= \sum_j y_j \frac{\partial M_{i,j}}{\partial x_i} \\
&= \sum_j y_j \left(M_{i,j} y_j - \beta \sum_{k \neq i} \frac{M_{i,j} - M_{k,j}}{x_i - x_k} \right) \\
&= \sum_j y_j^2 M_{i,j} - \beta \sum_j y_j \sum_{k \neq i} \frac{M_{i,j} - M_{k,j}}{x_i - x_k} \\
&= \sum_j y_j^2 M_{i,j} - \beta \sum_{k \neq i} \frac{\frac{\partial I}{\partial x_i} - \frac{\partial I}{\partial x_k}}{x_i - x_k}.
\end{aligned} \tag{4.14}
$$

If we take the sum over i, using equation (4.5), we get

$$\sum_i \frac{\partial^2 I}{\partial x_i^2} + \beta \sum_i \sum_{k\neq i} \frac{\frac{\partial I}{\partial x_i} - \frac{\partial I}{\partial x_k}}{x_i - x_k} = \sum_j y_j^2 I, \tag{4.15}$$

i.e. we recover the Calogero equation

$$\boxed{H_{\text{Calogero}}.I_{\beta,n} = \left(\sum_j y_j^2\right) I_{\beta,n}.} \tag{4.16}$$

4.5. Matrix form of the linear equations

The linear equations are n^2 linear equations of order 1, for n^2 unknown functions $M_{i,j}$. They can be summarized into a matricial equation,

$$MY = KM, \tag{4.17}$$

where K is a matricial operator,

$$K_{ii} = \frac{\partial}{\partial x_i} + \beta \sum_{k\neq i} \frac{1}{(x_i - x_k)}, \qquad K_{ik} = -\frac{\beta}{(x_i - x_k)} \quad i \neq k, \tag{4.18}$$

and more generally this implies

$$MY^p = K^p M \tag{4.19}$$

and therefore, for any polynomial P

$$M \cdot P(Y) = P(K) \cdot M. \tag{4.20}$$

In particular, if we choose the characteristic polynomial of Y,

$$0 = \prod_{i=1}^{n} (y_i - K) \cdot M. \tag{4.21}$$

Let us introduce the vector

$$e = (1, 1, \ldots, 1)^t. \tag{4.22}$$

It is such that M is a stochastic matrix, i.e.

$$M \cdot e = I_{\beta,n} e, \qquad e^t \cdot M = I_{\beta,n} e^t. \tag{4.23}$$

We thus have, for any polynomial P,

$$e^t P(K) e \cdot I_{\beta,n} = I_{\beta,n} \operatorname{Tr} P(Y). \tag{4.24}$$

Note that the Calogero equation is the case $P(K) = K^2$.

If P is the characteristic polynomial of Y, we get another differential equation for $I_{\beta,n}$:

$$\boxed{\forall\, i, \quad \sum_j \left(\prod_{l=1}^{n} (y_l - K)\right)_{i,j} \cdot I_{\beta,n} = 0.} \tag{4.25}$$

And if $P(K) = \prod_{l\neq j} (y_l - K)$, we get

$$\boxed{M_{i,j} = \sum_m \left(\prod_{l\neq j} \frac{y_l - K}{y_l - y_j}\right)_{i,m} \cdot I_{\beta,n}.} \tag{4.26}$$

This last relation allows us to reconstruct $M_{i,j}$ if we know $I_{\beta,n}$.

Finally, before leaving this section, we just mention that those operators $K_{i,j}$ are also related to the Laplacian over the set of matrices $E_{\beta,n}$, as was noted recently by Zuber [36].

4.6. Linear equation from loop equations

There is another way of deriving those Dunkl-like linear equations for the angular integrals, using loop equations of an associated 2-matrix model.

Consider the following 2-matrix integral, where M_1 and M_2 are both in the $E_{\beta,n}$ ensemble:

$$Z = \int \mathrm{d}M_1\, \mathrm{d}M_2\, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}. \tag{4.27}$$

After diagonalization of $M_1 = O_1 X O_1^{-1}$ and $M_2 = O_2 Y O_2^{-1}$, we have

$$Z = \int \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}O_1\, \mathrm{d}O_2 \Delta(X)^{2\beta}\Delta(Y)^{2\beta}\, \mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}\, \mathrm{e}^{\operatorname{Tr} X O_1^{-1} O_2 Y O_2^{-1} O_1}. \tag{4.28}$$

We redefine $O_2 = O_1.O$, and the integral over O_1 gives 1, and the integral over O gives the angular integral

$$Z = \int \mathrm{d}X\, \mathrm{d}Y \Delta(X)^{2\beta}\Delta(Y)^{2\beta}\, \mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))} I_{\beta,n}(X, Y). \tag{4.29}$$

We can do a similar change of variable for moments:

$$\begin{aligned}
\left\langle \operatorname{Tr} \frac{1}{x-M_1}\frac{1}{y-M_2}\right\rangle &= \frac{1}{Z}\int \mathrm{d}M_1 \mathrm{d}M_2\, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)} \operatorname{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\\
&= \frac{1}{Z}\int \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}O_1\, \mathrm{d}O_2 \Delta(X)^{2\beta}\Delta(Y)^{2\beta}\, \mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}\\
&\quad \times \mathrm{e}^{\operatorname{Tr} X O_1^{-1} O_2 Y O_2^{-1} O_1} \operatorname{Tr}\left(\frac{1}{x-X} O_1^{-1} O_2 \frac{1}{y-Y} O_2^{-1} O_1\right)\\
&= \frac{1}{Z}\int \mathrm{d}X\, \mathrm{d}Y \Delta(X)^{2\beta}\Delta(Y)^{2\beta}\, \mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}\\
&\quad \times \sum_{i,j} M_{i,j}(X, Y)\frac{1}{x-x_i}\frac{1}{y-y_j},
\end{aligned} \tag{4.30}$$

where $M_{i,j}(X, Y)$ is the Morozov moment defined in equation (4.1).

Loop equations amount to saying that an integral is invariant under a change of variables. Thus, we change $M_1 \to M_1 + \epsilon \frac{1}{x-M_1}\frac{1}{y-M_2} + O(\epsilon^2)$ in Z, and to order 1 in ϵ we get (the loop equations for $\beta = 1/2, 1, 2$ ensembles can be found in [3] and the Jacobian is easily computed in eigenvalue representation; see appendix B, equation (B.8))

$$\begin{aligned}
0 = &\left\langle \operatorname{Tr} \frac{1}{x-M_1}\frac{M_2}{y-M_2}\right\rangle - \left\langle \operatorname{Tr} \frac{V_1'(M_1)}{x-M_1}\frac{1}{y-M_2}\right\rangle + \beta \left\langle \operatorname{Tr} \frac{1}{x-M_1} \operatorname{Tr} \frac{1}{x-M_1}\frac{1}{y-M_2}\right\rangle\\
&+ (\beta-1)\frac{\partial}{\partial x}\left\langle \operatorname{Tr} \frac{1}{x-M_1}\frac{1}{y-M_2}\right\rangle,
\end{aligned} \tag{4.31}$$

i.e., going to eigenvalues $M_1 = O_1 X O_1^{-1}$ and $M_2 = O_2 Y O_2^{-1}$

$$\begin{aligned}
0 = &\sum_{i,j}\left\langle \frac{(y_j - V_1'(x_i))M_{i,j}(X, Y)}{(x-x_i)(y-y_j)}\right\rangle + \beta \sum_{i\neq l}\sum_j \left\langle \frac{M_{i,j}(X, Y)}{(x-x_l)(x-x_i)(y-y_j)}\right\rangle\\
&+ \sum_{i,j}\left\langle \frac{M_{i,j}(X, Y)}{(x-x_i)^2(y-y_j)}\right\rangle.
\end{aligned} \tag{4.32}$$

The last term can be integrated by parts:

$$\sum_{i,j}\left\langle \frac{M_{i,j}(X,Y)}{(x-x_i)^2(y-y_j)}\right\rangle = \sum_{i,j}\int \mathrm{d}X\,\mathrm{d}Y\,\Delta(X)^{2\beta}\Delta(Y)^{2\beta}\,\mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}$$

$$\times M_{i,j}(X,Y)\frac{\partial}{\partial x_i}\frac{1}{(x-x_i)(y-y_j)}$$

$$= -\sum_{i,j}\int \mathrm{d}X\,\mathrm{d}Y\frac{1}{(x-x_i)(y-y_j)}$$

$$\times \frac{\partial}{\partial x_i}M_{i,j}(X,Y)\Delta(X)^{2\beta}\Delta(Y)^{2\beta}\,\mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}$$

$$= \sum_{i,j}\int \mathrm{d}X\,\mathrm{d}Y\frac{M_{i,j}(X,Y)}{(x-x_i)(y-y_j)}\Delta(X)^{2\beta}\Delta(Y)^{2\beta}$$

$$\times\,\mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}\left(V_1'(x_i)-\sum_{l\neq i}\frac{2\beta}{x_i-x_l}\right)$$

$$-\sum_{i,j}\int \mathrm{d}X\,\mathrm{d}Y\frac{1}{(x-x_i)(y-y_j)}\Delta(X)^{2\beta}\Delta(Y)^{2\beta}$$

$$\times\,\mathrm{e}^{-\operatorname{Tr}(V_1(X)+V_2(Y))}\frac{\partial M_{i,j}(X,Y)}{\partial x_i}$$

$$= \sum_{i,j}\left\langle\frac{V_1'(x_i)M_{i,j}(X,Y)}{(x-x_i)(y-y_j)}\right\rangle - 2\beta\sum_{l\neq i}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x_i-x_l)(x-x_i)(y-y_j)}\right\rangle$$

$$-\sum_{i,j}\left\langle\frac{1}{(x-x_i)(y-y_j)}\frac{\partial M_{i,j}(X,Y)}{\partial x_i}\right\rangle. \tag{4.33}$$

Therefore, we have

$$\sum_{i,j}\left\langle\frac{1}{(x-x_i)(y-y_j)}\frac{\partial M_{i,j}(X,Y)}{\partial x_i}\right\rangle$$

$$= \sum_{i,j}\left\langle\frac{y_jM_{i,j}(X,Y)}{(x-x_i)(y-y_j)}\right\rangle - 2\beta\sum_{l\neq i}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x_i-x_l)(x-x_i)(y-y_j)}\right\rangle$$

$$+\beta\sum_{i\neq l}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x-x_l)(x-x_i)(y-y_j)}\right\rangle$$

$$= \sum_{i,j}\left\langle\frac{y_jM_{i,j}(X,Y)}{(x-x_i)(y-y_j)}\right\rangle - 2\beta\sum_{l\neq i}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x_i-x_l)(x-x_i)(y-y_j)}\right\rangle$$

$$+\beta\sum_{i\neq l}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x-x_l)(x_l-x_i)(y-y_j)}\right\rangle$$

$$+\beta\sum_{i\neq l}\sum_j\left\langle\frac{M_{i,j}(X,Y)}{(x-x_i)(x_i-x_l)(y-y_j)}\right\rangle$$

$$= \sum_{i,j} \left\langle \frac{y_j M_{i,j}(X,Y)}{(x-x_i)(y-y_j)} \right\rangle$$

$$- \beta \sum_{l \neq i} \sum_{j} \left\langle \frac{1}{(x-x_i)(y-y_j)} \frac{M_{i,j}(X,Y) - M_{l,j}(X,Y)}{(x_i - x_l)} \right\rangle. \tag{4.34}$$

Since this equation must hold for any V_1 and V_2, x, y, i.e. for any measure on X and Y, it must hold term by term, i.e., we recover the linear equation

$$\frac{\partial M_{i,j}(X,Y)}{\partial x_i} = y_j M_{i,j}(X,Y) - \beta \sum_{l \neq i} \frac{M_{i,j}(X,Y) - M_{l,j}(X,Y)}{(x_i - x_l)}. \tag{4.35}$$

Of course, the loop equation coming from the change of variable $M_2 \to M_2 + \epsilon \frac{1}{x-M_1} \frac{1}{y-M_2} + O(\epsilon^2)$ gives the symmetric linear equation:

$$\frac{\partial M_{i,j}(X,Y)}{\partial y_j} = x_i M_{i,j}(X,Y) - \beta \sum_{l \neq j} \frac{M_{i,j}(X,Y) - M_{i,l}(X,Y)}{(y_j - y_l)} \qquad \text{QED.} \tag{4.36}$$

5. Principal terms and the τ_{ij} variables

As we mentioned in the introduction, it was noted, in particular by Brezin and Hikami [6], that the angular integral can be written as combinations of exponential terms, and polynomials (for the β integer, series otherwise), of some reduced variables $\tau_{i,j} = -\frac{1}{2}(x_i - x_j)(y_i - y_j)$. Here, we show how our recursion gives such a form.

We thus define

Definition 5.1. *We define the principal term $\hat{\mathcal{I}}_{\beta,n}(X;Y)$ from the recursion,*

$$\hat{\mathcal{I}}_{\beta,1} = 1$$

and

$$\hat{\mathcal{I}}_{\beta,n}(X_n;Y_n) = \Delta(Y_n)^{2\beta} \prod_{i=1}^{n-1} (x_i - x_n) \operatorname{Res}_{\lambda_i \to y_i} \frac{d\lambda_1}{(\lambda_1 - y_1)^\beta} \cdots \frac{d\lambda_{n-1}}{(\lambda_{n-1} - y_{n-1})^\beta}$$

$$\times \frac{\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1},\Lambda)\, e^{\sum_i (x_i - x_n)(\lambda_i - y_i)}}{\prod_{k=1}^{n} \prod_{i=1,\neq k}^{n-1} (y_k - \lambda_i)^\beta}. \tag{5.1}$$

It is such that (the sum over permutations comes from the sum of residues at all poles in recursion equation (3.31) of theorem 3.3)

$$\boxed{I_{\beta,n}(X,Y) = \sum_{\sigma} \frac{e^{\sum_{i=1}^n x_i y_{\sigma(i)}}}{\Delta(X)^{2\beta} \Delta(Y_\sigma)^{2\beta}} \hat{\mathcal{I}}_{\beta,n}(X,Y_\sigma).} \tag{5.2}$$

In [6], Brezin and Hikami observed and conjectured that when β is an integer, $\hat{\mathcal{I}}_{\beta,n}(X,Y)$ is a polynomial in the variables $\tau_{i,j} = -\frac{1}{2}(x_i - x_j)(y_i - y_j)$.

For instance, if $\beta = 1$, we have for arbitrary n

$$\hat{\mathcal{I}}_{1,n} = \prod_{i<j} \tau_{i,j}. \tag{5.3}$$

And, if $n = 2$, we have for arbitrary β

$$\begin{aligned}\hat{\mathcal{I}}_{\beta,2} &= (x_1 - x_2)(y_1 - y_2)^{2\beta}\,\mathrm{Res}_{\lambda\to 0}\frac{\mathrm{d}\lambda}{\lambda^\beta}\frac{\mathrm{e}^{(x_1-x_2)\lambda}}{(y_2 - y_1 - \lambda)^\beta}\\ &= (x_1 - x_2)(y_1 - y_2)^{2\beta}\frac{\partial^{\beta-1}}{\partial\lambda^{\beta-1}}\left(\frac{\mathrm{e}^{(x_1-x_2)\lambda}}{(y_2 - y_1 - \lambda)^\beta}\right)_{\lambda=0}\\ &= (x_1 - x_2)(y_1 - y_2)^{2\beta}\sum_{k=0}^{\beta-1}\frac{(\beta-1)!}{k!(\beta-1-k)!}(x_1 - x_2)^{\beta-1-k}(y_2 - y_1)^{-\beta-k}\frac{(\beta-1+k)!}{(\beta-1)!}\\ &= \sum_{k=0}^{\beta-1}\frac{(\beta-1+k)!}{k!(\beta-1-k)!}\,((x_1 - x_2)(y_2 - y_1))^{\beta-k}\\ &= 2^\beta Q_{\beta,0}(\tau_{1,2}),\end{aligned} \tag{5.4}$$

i.e. we recover the well-known result that $\hat{\mathcal{I}}_{\beta,2}$ is the Bessel polynomial of degree β.

We are going to prove the conjecture of Brezin–Hikami for all n and for all $\beta \in \mathbb{N}$, but first, let us prove some preliminary properties.

Lemma 5.1. *$\hat{\mathcal{I}}_{\beta,n}$ is a polynomial in all variables x_i and y_j, and it is symmetric under the exchange $X \leftrightarrow Y$ and under the permutation of pairs $(x_i, y_i) \leftrightarrow (x_j, y_j)$, and under translations $X \to X + \mathrm{cte.Id}$ or $Y \to Y + \mathrm{cte.Id}$.*

Proof. If β is an integer, the recursion relation (5.1) leads to (we write $x_{i,j} = x_i - x_j$, $y_{i,j} = y_i - y_j$)

$$\hat{\mathcal{I}}_{\beta,n+1} = \prod_{i=1}^{n} x_{i,n+1}y_{n+1,i}^{\beta}\left(\frac{\partial}{\partial\lambda_i}\right)^{\beta-1}\left[\mathrm{e}^{x_{i,n+1}\lambda_i}\prod_{1\leqslant j\leqslant n+1, j\neq i}\frac{y_{j,i}^{\beta}}{(y_{j,i} - \lambda_i)^\beta}\right]\hat{\mathcal{I}}_{\beta,n}(X_n, Y_n + \Lambda)|_{\Lambda=0}, \tag{5.5}$$

which shows by recursion that $\hat{\mathcal{I}}_{\beta,n+1}$ is a rational function of all x_i's and y_j's.

We know, from its very definition that the angular integral,

$$I_{\beta,n}(X, Y) = \sum_\sigma \frac{\mathrm{e}^{\sum_{i=1}^n x_i y_{\sigma(i)}}}{\Delta(X)^{2\beta}\Delta(Y_\sigma)^{2\beta}}\hat{\mathcal{I}}_{\beta,n}(X, Y_\sigma), \tag{5.6}$$

is symmetric in all x_i's and y_j's, and in the exchange $X \leftrightarrow Y$. Since the exponentials are linearly independent on the ring of rational functions, each term must be symmetric in permutations of pairs (x_i, y_i)'s, i.e. $\hat{\mathcal{I}}_{\beta,n}(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the pairs (x_i, y_i)'s, and also symmetric under $X \leftrightarrow Y$.

Moreover, $\hat{\mathcal{I}}_{\beta,n+1}$ is clearly a polynomial in the variables x_{n+1} and y_{n+1}, and because of the symmetry, it must also be a polynomial in all variables. Translation invariance is also clear from the recursion formula. □

Theorem 5.1. *(Conjecture of Brezin and Hikami)*

$\hat{\mathcal{I}}_{\beta,n}$ is a symmetric polynomial of degree β in the $\tau_{i,j}$'s.

Proof. Using lemma 5.1, it is easy to see that $\hat{\mathcal{I}}_{\beta,n}$ fulfils the hypothesis of lemma A.3 in appendix A, and this proves the theorem. □

That is, we have proved the conjecture of Brezin and Hikami [6]. In fact, we note that the property of being a polynomial in the τ's is not specific to angular integrals, but comes only from the global symmetries.

5.1. Recursion without residues for β integer

For $\beta \in \mathbb{N}$, the residues in recursion equation (5.1) can be performed, and they compute derivatives of the integrand. Thus equation (5.1) can be rewritten:

$$\begin{aligned}\hat{\mathcal{I}}_{\beta,n}(X_n;Y_n) &= \Delta(Y_n)^{2\beta}\prod_{i=1}^{n-1}(x_i-x_n)\,\mathrm{Res}_{\,\lambda_i\to a_i}\frac{\mathrm{d}\lambda_1}{(\lambda_1-a_1)^\beta}\cdots\frac{\mathrm{d}\lambda_{n-1}}{(\lambda_{n-1}-a_{n-1})^\beta}\\ &\quad\times\left.\frac{\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1},\Lambda)\,\mathrm{e}^{\sum_i(x_i-x_n)(\lambda_i-y_i)}}{\prod_{k=1}^n\prod_{i=1,\neq k}^{n-1}(y_k-\lambda_i)^\beta}\right|_{a_i=y_i}\\ &= \frac{\Delta(Y_n)^{2\beta}}{(\beta-1)!^{n-1}}\prod_{i=1}^{n-1}(x_i-x_n)\prod_i\left(\frac{\partial}{\partial a_i}\right)^{\beta-1}\mathrm{Res}_{\,\lambda_i\to a_i}\frac{\mathrm{d}\lambda_1}{(\lambda_1-a_1)}\cdots\frac{\mathrm{d}\lambda_{n-1}}{(\lambda_{n-1}-a_{n-1})}\\ &\quad\times\left.\frac{\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1},\Lambda)\,\mathrm{e}^{\sum_i(x_i-x_n)(\lambda_i-y_i)}}{\prod_{k=1}^n\prod_{i=1,\neq k}^{n-1}(y_k-\lambda_i)^\beta}\right|_{a_i=y_i},\end{aligned}\tag{5.7}$$

i.e. we can perform the residues:

$$\boxed{\hat{\mathcal{I}}_{\beta,n}(X_n;Y_n) = \frac{\Delta(Y_n)^{2\beta}}{(\beta-1)!^{n-1}}\prod_{i=1}^{n-1}x_{i,n}\left(\frac{\partial}{\partial a_i}\right)^{\beta-1}\left.\frac{\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1},a)\,\mathrm{e}^{\sum_i x_{i,n}(a_i-y_i)}}{\prod_{k=1}^n\prod_{i=1,\neq k}^{n-1}(y_k-a_i)^\beta}\right|_{a_i=y_i}.}\tag{5.8}$$

More explicitly,

$$\begin{aligned}\hat{\mathcal{I}}_{\beta,n}(X_n;Y_n) &= \prod_{i=1}^{n-1}\sum_{\gamma_i,\beta_{i,k}=0}^{\beta-1}\frac{(x_{i,n}y_{n,i})^{\beta_{i,n}}}{\Gamma(\beta-\gamma_i-\sum_k\beta_{i,k})\beta_{i,n}!}\prod_{k=1,k\neq i}^{n-1}\frac{\Gamma(\beta+\beta_{i,k})}{\beta_{i,k}!\Gamma(\beta)}\frac{1}{(x_{i,n}y_{k,i})^{\beta_{i,k}}}\\ &\quad\times\frac{1}{\gamma_i!}\left(\frac{1}{x_{i,n}}\frac{\partial}{\partial a_i}\right)^{\gamma_i}\hat{\mathcal{I}}_{\beta,n-1}(X_{n-1},a)|_{a_i=y_i}.\end{aligned}\tag{5.9}$$

5.2. $n=3$

For $n=3$, and arbitrary β, we prove that

Theorem 5.2.

$$\boxed{\hat{\mathcal{I}}_{\beta,3} = \sum_{k=0}^{\infty}\frac{\Gamma(\beta-k)}{2^{3k}k!\Gamma(\beta+k)}\prod_{i<j}Q_{\beta,k}(\tau_{ij}).}\tag{5.10}$$

In fact, for the β integer, the sum over k is finite and reduces to $k\leqslant\beta-1$.

This expression was obtained in [4] for $\beta=1/2$.

The proof, which is rather technical, is given in appendix C. We used the Calogero equation.

5.3. Conjecture for higher n

Applying the recursion relations of this paper, we also computed the $n=4$ case for small values of β:

$$\hat{\mathcal{I}}_{2,4} = \prod_{i<j} Q_{2,0}(\tau_{ij}) + \frac{1}{16} Q_{2,0}(\tau_{1,2}) Q_{2,0}(\tau_{1,3}) Q_{2,0}(\tau_{1,4}) Q_{2,1}(\tau_{2,3}) Q_{2,1}(\tau_{2,4}) Q_{2,1}(\tau_{3,4}) + \mathrm{sym}$$
$$+ \frac{1}{128} Q_{2,0}(\tau_{1,2}) Q_{2,1}(\tau_{1,3}) Q_{2,1}(\tau_{1,4}) Q_{2,1}(\tau_{2,3}) Q_{2,1}(\tau_{2,4}) Q_{2,1}(\tau_{3,4}) + \mathrm{sym} \tag{5.11}$$

and

$$\hat{\mathcal{I}}_{3,4} = 64 \prod_{i<j} Q_{3,0}(\tau_{ij}) + \frac{4}{3} Q_{3,0}(\tau_{1,2}) Q_{3,0}(\tau_{1,3}) Q_{3,0}(\tau_{1,4}) Q_{3,1}(\tau_{2,3}) Q_{3,1}(\tau_{2,4}) Q_{3,1}(\tau_{3,4}) + \mathrm{sym}$$
$$+ \frac{1}{48} Q_{3,0}(\tau_{1,2}) Q_{3,0}(\tau_{1,3}) Q_{3,0}(\tau_{1,4}) Q_{3,2}(\tau_{2,3}) Q_{3,2}(\tau_{2,4}) Q_{3,2}(\tau_{3,4}) + \mathrm{sym}$$
$$+ \cdots . \tag{5.12}$$

These expressions lead us to conjecture a general form in terms of Bessel polynomials.

Conjecture 5.1. *We conjecture that for all n and β, $\hat{\mathcal{I}}_{\beta,n}$ is of the form*

$$\boxed{\hat{\mathcal{I}}_{\beta,n} = \sum_{\{l\}} A_{\{l\}} \prod_{i<j} Q_{\beta,l_{(i,j)}}(\tau_{i,j}).} \tag{5.13}$$

Unfortunately, we have not been able so far to determine the general form of the coefficients $A_{\{l\}}$ for $n > 3$ (except $n = 4$ and $\beta = 2$).

5.4. *The symplectic case $\beta = 2$*

The case $\beta = 2$ is that of the symplectic group. It is, in principle, dual to the orthogonal group $\beta = 1/2$, which was more studied [2, 4], but in fact it should be easier to treat with our approach since $\beta = 2$ is an integer.

For $\beta = 2$, the recursion equation (5.8) reduces to

$$\hat{\mathcal{I}}_{2,n} = \Delta(Y_n)^4 \prod_{i=1}^{n-1} x_{i,n} \prod_i \frac{\partial}{\partial a_i} \frac{\hat{\mathcal{I}}_{2,n-1}(X_{n-1}, a)\, \mathrm{e}^{\sum_i (x_i - x_n)(a_i - y_i)}}{\prod_{k=1}^{n} \prod_{i=1,\neq k}^{n-1} (y_k - a_i)^2} \Bigg|_{a_i = y_i}$$
$$= \prod_{i=1}^{n-1} x_{i,n} y_{i,n}^2 \left(x_i - x_n - \sum_{k=1,\neq i}^{n} \frac{2}{y_i - y_k} + \frac{\partial}{\partial a_i} \right) \hat{\mathcal{I}}_{2,n-1}(X_{n-1}, a)|_{a_i = y_i}. \tag{5.14}$$

From a recursion hypothesis, we assume that $\hat{\mathcal{I}}_{2,n-1}(X_{n-1}, a)$ is a polynomial in the τ's of the form

$$\hat{\mathcal{I}}_{2,n-1}(X_{n-1}, Y_{n-1}) = \sum_{\{l\}} A_{\{l\}} \prod_{i<j} Q_{2,l_{i,j}}(\tau_{i,j}), \tag{5.15}$$

where for every pair (i, j) we have $l_{i,j} \in \{0, 1\}$. We recall that

$$|0\rangle_\tau = Q_{2,0}(\tau) = \tau^2 + \tau, \qquad |1\rangle_\tau = Q_{1,0}(\tau) = 2\tau. \tag{5.16}$$

Thus we may write

$$\frac{\partial}{\partial a_i} = -\frac{1}{2} \sum_{k\neq i} x_{i,k} \frac{\partial}{\partial t_{i,k}}, \qquad t_{i,k} = -\frac{1}{2}(x_i - x_k)(a_i - a_k), \tag{5.17}$$

and we define the operators $C_{i,k}$ acting on functions of the variable $\tau_{i,k}$ such that

$$C_{i,k} = \frac{1}{\tau_{i,k}} - \frac{1}{2} \frac{\partial}{\partial t_{i,k}}, \tag{5.18}$$

and all derivatives must be eventually computed at $t_{i,k} = \tau_{i,k}$.

Since our operators act on expressions of the form of equation (5.15), we need to compute

$$C_{i,j}|0\rangle = C_{i,j}.Q_{2,0}(\tau_{i,j}) = C_{i,j}.\big(\tau_{i,j}^2 + \tau_{i,j}\big) = \tfrac{1}{2}, \tag{5.19}$$

$$C_{i,j}|1\rangle = C_{i,j}.Q_{2,1}(\tau_{i,j}) = C_{i,j}.2\tau_{i,j} = 1. \tag{5.20}$$

And we may also have terms of the form $C_{i,j}.C_{j,i}$, for which we have

$$C_{i,j}C_{j,i}|0\rangle = C_{i,j}C_{j,i}.(\tau_{i,j}^2 + \tau_{i,j}) = -\tfrac{1}{2}, \qquad C_{i,j}C_{j,i}|1\rangle = C_{i,j}C_{j,i}.\tau_{i,j} = 0. \tag{5.21}$$

Finally, we have

$$\hat{\mathcal{I}}_{2,n} = \prod_{i=1}^{n-1} \tau_{i,n}^2 \prod_{i=1}^{n-1} \left(1 + \frac{1}{\tau_{i,n}} + \sum_{k=1,\neq i}^{n-1} \frac{x_{i,k}}{x_{i,n}} C_{i,k}\right) \hat{\mathcal{I}}_{2,n-1}|_{t_{i,j}=\tau_{i,j}}. \tag{5.22}$$

It is more convenient to rewrite this in terms of a Hilbert space with basis $|0\rangle = Q_{2,0}$ and $|1\rangle = Q_{2,1}$, and thus

$$\hat{\mathcal{I}}_{2,n} = \prod_{i=1}^{n-1} \left(\tau_{i,n}^2 + \tau_{i,n} + \frac{1}{2}\tau_{i,n} \sum_{k=1,\neq i}^{n-1} \frac{y_{n,i}}{y_{k,i}} A_{i,k}\right) \hat{\mathcal{I}}_{2,n-1}, \tag{5.23}$$

where

$$A_{i,k} = 2\tau_{i,k} C_{i,k}. \tag{5.24}$$

We have

$$A|0\rangle = \tfrac{1}{2}|1\rangle, \qquad A|1\rangle = |1\rangle \tag{5.25}$$

and

$$A_{i,k}A_{k,i} = 2(A_{i,k} - 1). \tag{5.26}$$

Unfortunately, we have not been able to go further with this formulation.

5.4.1. Triangle conjecture. We have seen that there is an operator formalism for computing angular integrals, with operators $A_{i,j}$ associated with 'edges' (i, j). However, it can be seen for $n = 3, 4$, that operator edges appear only in certain combinations, which involve triangles (i, j, k). We thus introduce the triangle operators

$$T_{i,j,k} = \pi_{i,j}\pi_{j,k}\pi_{k,i}, \tag{5.27}$$

where $\pi_{i,j} = \pi_{j,i}$ is the projector on state $\frac{1}{2}|1\rangle_{i,j} = \tau_{i,j}$:

$$\pi_{i,j}.\big(\tau_{i,j}^2 + \tau_{i,j}\big) = \tau_{i,j}, \qquad \pi_{i,j}.\tau_{i,j} = \tau_{i,j}, \qquad \pi_{i,j} = \pi_{j,i}, \qquad \pi_{i,j}^2 = \pi_{i,j}. \tag{5.28}$$

With this notation we have

$$\hat{\mathcal{I}}_{2,3} = \left(1 + \frac{1}{2}T_{1,2,3}\right) . \prod_{1\leqslant i<j\leqslant 3} |0\rangle_{i,j}, \tag{5.29}$$

where we recall that $|0\rangle_{i,j} = \tau_{i,j}^2 + \tau_{i,j}$.

And

$$\hat{\mathcal{I}}_{2,4} = \Bigg(1 + \frac{1}{2}(T_{1,2,3} + T_{1,2,4} + T_{1,3,4} + T_{2,3,4}) + \frac{1}{4}(T_{1,2,3}T_{1,2,4} + T_{1,2,3}T_{1,3,4} + T_{1,2,4}T_{1,3,4} + T_{1,2,3}T_{2,3,4} + T_{1,2,4}T_{2,3,4} + T_{1,3,4}T_{2,3,4})\Bigg) \cdot \prod_{1\leqslant i<j\leqslant 4} |0\rangle_{i,j}. \tag{5.30}$$

We are naturally led to conjecture that

$$\hat{\mathcal{I}}_{2,n} = \left(\sum_{\text{triangulations } \mathcal{T}} C_{\mathcal{T}} \prod_{T \in \mathcal{T}} \right) . \prod_{i<j} |0\rangle_{i,j}. \tag{5.31}$$

We have

$$C_{\emptyset} = 1, \qquad C_{(i,j,k)} = \tfrac{1}{2}, \qquad C_{(i,j,k),(i,j,l)} = \tfrac{1}{4}, \qquad \ldots. \tag{5.32}$$

We have not been able so far to prove this conjecture. It is to be noted from the low values of n that triangles seem to play a role for all $\beta \in \mathbb{N}$.

5.4.2. Additional results: determinantal recursion. Just for completeness, we give another form of the recursion equation (5.14) in terms of determinants:

$$\hat{I}_{2,n+1}(X_{n+1}; Y_{n+1}) = \frac{\det\left[X - x_{n+1} - \frac{2}{Y - y_{n+1}} + B + \partial_Y\right]}{\det(X - x_{n+1})} \hat{I}_{2,n}(X_n; Y_n), \tag{5.33}$$

where

$$\hat{I}_{2,n}(X, Y) = \frac{1}{\prod_{i<j} \tau_{i,j}^2} \hat{\mathcal{I}}_{2,n}(X, Y) \tag{5.34}$$

and where B is the anti-symmetric matrix,

$$B_{ij} = \frac{\sqrt{2}}{y_i - y_j}, \tag{5.35}$$

and $\partial_Y = \operatorname{diag}(\partial_{y_1}, \ldots, \partial_{y_n})$. This is proved by observing that the expansion of the determinant equation (5.33) can be interpreted like a Wick's expansion equivalent to equation (5.14).

6. Conclusion

In this paper, we have found many new relations and new representations of angular integrals.

First, we have been able to rewrite angular integrals with a complicated Haar measure, in terms of usual Lebesgue measure contour integrals. Then, we have deduced duality and recursion formulae.

This allowed us to prove Brezin–Hikami's conjecture and to find some explicit form for $n = 3$, and conjecture some explicit form in terms of Bessel polynomials for the general case.

For $\beta = 2$, we have simplified our recursion (computed the residues). The same method seems to be applicable for higher β, but we have not done it in this paper.

We have obtained many new forms of angular integrals, but unfortunately, this does not seem to be the end of the story. Our expressions are still not explicit enough to be useful for computing matrix integrals. The form of our expressions strongly suggests that the kernel-determinantal formulae [29] in the $\beta = 1$ case could be replaced by hyperdeterminantal formulae for higher β, but this is still to be understood. The best thing would be to get expressions with enough structure to generalize the method over integration of matrix variables of Mehta [30].

Acknowledgments

We would like to thank G Akemann, E Brezin, P Desrosiers, S Hikami, A Prats-Ferrer and J B Zuber for useful and fruitful discussions on this subject. This work is partly supported by the Enigma European network MRT-CT-2004-5652, by the ANR project Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01, by the Enrage European network MRTN-CT-2004-005616, by the European Science Foundation through the Misgam program, by the French and Japanese governments through PAI Sakurav and by the Quebec government with the FQRNT.

Appendix A. Polynomials of τ

Lemma A.1. *Let*

$$P_n(X,Y) = x_1x_2\ldots x_{n-1}x_ny_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n} + y_1y_2\ldots y_{n-1}y_nx_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n}. \tag{A.1}$$

We prove that P_n is a polynomial of degree n in the τ's, where $\tau_{i,2n+1} = -\frac{1}{2}x_iy_i$ and $\tau_{i,j} = -\frac{1}{2}(x_i - x_j)(y_i - y_j)$, with integer coefficients

$$P_n \in \mathbb{Z}[\tau]. \tag{A.2}$$

Proof. It clearly holds for $n = 0$ and $n = 1$. Indeed, the $n = 1$ case reads

$$x_1y_2 + x_2y_1 = x_1y_1 + x_2y_2 - (x_1 - x_2)(y_1 - y_2) = 2(\tau_{1,2} - \tau_{1,3} - \tau_{2,3}). \tag{A.3}$$

Assume that the lemma holds up to $n - 1$, and let us prove it for n. In the following, $A \equiv B$ means that $A - B$ is a polynomial in the τ's:

$$\begin{aligned}
P_n &= x_1x_2\ldots x_{n-1}x_ny_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n} + y_1y_2\ldots y_{n-1}y_nx_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n}\\
&= (x_1y_{n+1} + x_{n+1}y_1)(x_2\ldots x_{n-1}x_ny_{n+2}\ldots y_{2n-1}y_{2n} + y_2\ldots y_{n-1}y_nx_{n+2}\ldots x_{2n-1}x_{2n})\\
&\quad - x_2\ldots x_{n-1}x_nx_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n}y_1 - y_2\ldots y_{n-1}y_ny_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n}x_1\\
&\equiv -x_2\ldots x_{n-1}x_nx_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n}y_1 - y_2\ldots y_{n-1}y_ny_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n}x_1
\end{aligned} \tag{A.4}$$

and then

$$\begin{aligned}
P_n &\equiv -x_2\ldots x_{n-1}x_nx_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n}y_1 - y_2\ldots y_{n-1}y_ny_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n}x_1\\
&\equiv -(x_2y_1 + x_1y_2)(x_3\ldots x_{n-1}x_nx_{n+1}y_{n+2}\ldots y_{2n-1}y_{2n} + y_3\ldots y_{n-1}y_ny_{n+1}x_{n+2}\ldots x_{2n-1}x_{2n})\\
&\quad + x_{n+1}x_1x_3\ldots x_{n-1}x_ny_2y_{n+2}\ldots y_{2n} + y_{n+1}y_1y_3\ldots y_nx_2x_{n+2}\ldots x_{2n}.
\end{aligned} \tag{A.5}$$

Repeating the same operation recursively we obtain $\forall k$:

$$\begin{aligned}
P_n &\equiv x_{n+1}\ldots x_{n+k}x_1x_{k+2}\ldots x_ny_2\ldots y_{k+1}y_{k+n+1}\ldots y_{2n}\\
&\qquad + y_{n+1}\ldots y_{n+k}y_1y_{k+2}\ldots y_nx_2\ldots x_{k+1}x_{k+n+1}\ldots x_{2n}.
\end{aligned} \tag{A.6}$$

In particular, for $k = n - 2$ we find

$$\begin{aligned}
P_n &\equiv x_{n+1}\ldots x_{2n-2}x_1x_ny_2\ldots y_{n-1}y_{2n-1}y_{2n} + y_{n+1}\ldots y_{2n-2}y_1y_nx_2\ldots x_{n-1}x_{2n-1}x_{2n}\\
&\equiv (x_1y_{2n-1} + x_{2n-1}y_1)(x_{n+1}\ldots x_{2n-2}x_ny_2\ldots y_{n-1}y_{2n}\\
&\quad + y_{n+1}\ldots y_{2n-2}y_nx_2\ldots x_{n-1}x_{2n})\\
&\quad - x_n\ldots x_{2n-1}y_1y_2\ldots y_{n-1}y_{2n} - y_n\ldots y_{2n-1}x_1x_2\ldots x_{n-1}x_{2n}),
\end{aligned} \tag{A.7}$$

and we repeat the same operations,

$$\begin{aligned}
P_n &\equiv -x_n\ldots x_{2n-1}y_1y_2\ldots y_{n-1}y_{2n} - y_n\ldots y_{2n-1}x_1x_2\ldots x_{n-1}x_{2n})\\
&\equiv -(x_ny_1 + x_1y_n)(x_{n+1}\ldots x_{2n-1}y_2\ldots y_{n-1}y_{2n} + y_{n+1}\ldots y_{2n-1}x_2\ldots x_{n-1}x_{2n})\\
&\quad + x_1x_{n+1}\ldots x_{2n-1}y_2\ldots y_ny_{2n} + y_1y_{n+1}\ldots y_{2n-1}x_2\ldots x_nx_{2n},
\end{aligned} \tag{A.8}$$

and once more

$$P_n \equiv x_1 x_{n+1} \dots x_{2n-1} y_2 \dots y_n y_{2n} + y_1 y_{n+1} \dots y_{2n-1} x_2 \dots x_n x_{2n}$$
$$\equiv (x_1 y_{2n} + x_{2n} y_1)(x_{n+1} \dots x_{2n-1} y_2 \dots y_n + y_{n+1} \dots y_{2n-1} x_2 \dots x_n)$$
$$- x_{n+1} \dots x_{2n} y_1 y_2 \dots y_n - y_{n+1} \dots y_{2n} x_1 \dots x_n. \tag{A.9}$$

Therefore, $P_n \equiv -P_n$, i.e. P_n is a polynomial in the τ's. □

Lemma A.2. *Let*

$$P_{\alpha,\beta}(X,Y) = \prod_{i=1}^{n} x_i^{\alpha_i} y_i^{\beta_i} + \prod_{i=1}^{n} x_i^{\beta_i} y_i^{\alpha_i}, \qquad \sum_i \alpha_i = \sum_i \beta_i = d. \tag{A.10}$$

We prove that $P_{\alpha,\beta}$ is a polynomial of degree d in the τ's, with integer coefficients

$$P_n \in \mathbb{Z}[\tau]. \tag{A.11}$$

Proof. We proceed by recursion on the total degree $d = \sum_i \alpha_i = \sum_i \beta_i$. The lemma clearly holds for $d = 0$ and $d = 1$.

Assume $d \geqslant 2$ and that the lemma holds up to $d - 1$. We will prove it for d.

- If there exists i such that $\alpha_i \beta_i > 0$, then $P_{\alpha,\beta}$ is the factor of $\tau_{i,n+1}$ times $P_{\alpha',\beta'}$ of smaller degree, and from the recursion hypothesis it holds.
- Assume that $\forall i, \alpha_i \beta_i = 0$. Since $d \geqslant 2$, there must exist some i such that $\alpha_i \geqslant 1$ and some j such that $\beta_j \geqslant 1$. Let us choose k and l such that

$$\alpha_k = \max_i \{\alpha_i\} \geqslant 1, \qquad \beta_l = \max_i \{\beta_i\} \geqslant 1. \tag{A.12}$$

We have $k \neq l$.

If $\alpha_k = \beta_l = 1$, then we can apply lemma A.1, and thus the lemma is proved for that case.

- Therefore, we now assume that $\alpha_k \beta_l \geqslant 2$, and with no loss of generality we may assume that $\alpha_k \geqslant 2$. We write

$$\begin{aligned} P_n &= x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_n^{\beta_n} + x_1^{\beta_1} \dots x_n^{\beta_n} y_1^{\alpha_1} \dots y_n^{\alpha_n} \\ &= (x_k y_l + x_l y_k)\big(x_1^{\alpha_1} \dots x_k^{\alpha_k - 1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_l^{\beta_l - 1} \dots y_n^{\beta_n} \\ &\quad + y_1^{\alpha_1} \dots y_k^{\alpha_k - 1} \dots y_n^{\alpha_n} x_1^{\beta_1} \dots x_l^{\beta_l - 1} \dots x_n^{\beta_n}\big) \\ &\quad - x_1^{\alpha_1} \dots x_k^{\alpha_k - 1} x_l^{\alpha_l + 1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_l^{\beta_l - 1} y_k^{\beta_k + 1} \dots y_n^{\beta_n} \\ &\quad - y_1^{\alpha_1} \dots y_k^{\alpha_k - 1} y_l^{\alpha_l + 1} \dots y_n^{\alpha_n} x_1^{\beta_1} \dots x_l^{\beta_l - 1} x_k^{\beta_k + 1} \dots x_n^{\beta_n} \\ &\equiv -x_k y_k \big(x_1^{\alpha_1} \dots x_k^{\alpha_k - 2} x_l^{\alpha_l + 1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_l^{\beta_l - 1} y_k^{\beta_k} \dots y_n^{\beta_n} \\ &\quad + y_1^{\alpha_1} \dots y_k^{\alpha_k - 2} y_l^{\alpha_l + 1} \dots y_n^{\alpha_n} x_1^{\beta_1} \dots x_l^{\beta_l - 1} x_k^{\beta_k} \dots x_n^{\beta_n}\big). \end{aligned} \tag{A.13}$$

From the recursion hypothesis, the RHS is a polynomial in the τ's, and thus we have proved the lemma. □

Lemma A.3. *Let P be a polynomial of $2n + 2$ variables $x_1, \dots, x_n, x_{n+1}, y_1, \dots, y_n, y_{n+1}$, with the following properties:*

- *P is invariant by translations $\forall i, x_i \to x_i + \delta x, y_i \to y_i + \delta y$,*
- *P is invariant under $\forall i, x_i \to \lambda x_i, y_i \to \frac{1}{\lambda} y_i$,*
- *P is symmetric in the exchange $X \leftrightarrow Y$.*

Then P is a polynomial of the $\tau_{i,j}$'s.

Proof. Because of invariance by translation, we can always assume that $x_{n+1} = y_{n+1} = 0$. Then, the other properties imply that P is a linear combination of monomials of the type $P_{\alpha,\beta}(X,Y)$ of lemma A.2. □

Appendix B. Loop equations

Loop equations for matrix models have been studied for a long time [31]. Loop equations (sometimes called Ward identities or Schwinger–Dyson equations) for β ensembles can be found, for instance, in [9, 16, 23–25, 35]. Here, we summarize the method.

- 1-matrix model in eigenvalue representation for arbitrary β.
Consider the integral

$$Z = \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \, \mathrm{e}^{-\sum_i V(\lambda_i)} \prod_{i<j} (\lambda_j - \lambda_i)^{2\beta}. \tag{B.1}$$

If we make an infinitesimal local change of variable $\lambda_i \to \lambda_i + \epsilon \lambda_i^k + O(\epsilon^2)$, we find

$$Z = (1 + O(\epsilon^2)) \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \prod_i \left(1 + \epsilon k \lambda_i^{k-1}\right) \mathrm{e}^{-\sum_i V(\lambda_i)} \prod_i \left(1 - \epsilon \lambda_i^k V'(\lambda_i)\right) \times \prod_{i<j} (\lambda_j - \lambda_i)^{2\beta} \prod_{i<j} \left(1 + 2\beta\epsilon \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j}\right), \tag{B.2}$$

i.e., by considering the term linear in ϵ

$$0 = \left\langle \sum_i k \lambda_i^{k-1} + \beta \sum_{l=0}^{k-1} \sum_{i \neq j} \lambda_i^l \lambda_j^{k-1-l} - \sum_i \lambda_i^k V'(\lambda_i) \right\rangle. \tag{B.3}$$

It can be written collectively by summing over k with $1/x^{k+1}$; this is equivalent to considering a local change of variable $\lambda_i \to \lambda_i + \frac{\epsilon}{x-\lambda_i} + O(\epsilon^2)$, and we write $\omega(x) = \sum_i \frac{1}{x-\lambda_i}$:

$$0 = \left\langle -\omega'(x) + \beta(\omega(x)^2 + \omega'(x)) - \sum_i \frac{V'(\lambda_i)}{x - \lambda_i} \right\rangle, \tag{B.4}$$

i.e.

$$0 = -\left\langle \operatorname{Tr} \frac{V'(M)}{x - M} \right\rangle + \beta \left\langle \operatorname{Tr} \frac{1}{x - M} \operatorname{Tr} \frac{1}{x - M} \right\rangle + (\beta - 1) \frac{\partial}{\partial x} \left\langle \operatorname{Tr} \frac{1}{x - M} \right\rangle. \tag{B.5}$$

- 2-matrix model for $\beta = 1/2, 1, 2$.
Similarly if we consider a 2-matrix model,

$$Z = \int \mathrm{d}M_1 \, \mathrm{d}M_2 \, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)}, \tag{B.6}$$

again we make a local change of variable $M_1 \to M_1 + \epsilon \frac{1}{x - M_1} A + O(\epsilon^2)$. The Jacobian of this change of variable is computed as a split rule (cf [3]), it can be computed for each of the three ensembles $\beta = 1/2, 1, 2$ and is:

$$\mathrm{d}M_1 \to \mathrm{d}M_1 \left(1 + \epsilon\beta \operatorname{Tr} \frac{1}{x - M_1} \operatorname{Tr} A \frac{1}{x - M_1} + \epsilon(\beta - 1) \frac{\partial}{\partial x} \operatorname{Tr} A \frac{1}{x - M_1} + O(\epsilon^2)\right). \tag{B.7}$$

Thus we find

$$0 = -\left\langle \operatorname{Tr}(V_1'(M_1) - M_2) \frac{1}{x - M_1} \right\rangle + \beta \left\langle \operatorname{Tr} \frac{1}{x - M_1} \operatorname{Tr} A \frac{1}{x - M_1} \right\rangle + (\beta - 1) \frac{\partial}{\partial x} \left\langle \operatorname{Tr} A \frac{1}{x - M_1} \right\rangle. \tag{B.8}$$

- Without a definition of a 2-matrix integral for arbitrary β, it is not possible to find the loop equation for any β. However, we see that equation (B.8) is valid for the 2-matrix model for $\beta = 1/2, 1, 2$, and is valid for the 1-matrix eigenvalue model for any β. Therefore, it is natural to take it as a definition of the 2-matrix model for arbitrary β.

Appendix C. Proof $n = 3$

C.1. Bessel zoology

We define the functions

$$Q_\beta(x) = \sum_{l=0}^{\infty} (-)^l \frac{\Gamma(\beta+l)}{\Gamma(\beta-l)} \frac{2^{-l}}{l!} x^{\beta-l}. \tag{C.1}$$

We have

$$\begin{aligned} 2x^2 Q'_\beta(x) - 2\beta x Q_\beta(x) &= -\sum_{l=1}^{\infty} (-)^l \frac{\Gamma(\beta+l)}{\Gamma(\beta-l)} \frac{2^{-l+1}}{(l-1)} x^{\beta-l+1}, \\ x^2 Q''_\beta(x) - 2\beta x Q'_\beta(x) + 2\beta Q_\beta(x) &= -\sum_{l=0}^{\infty} (-)^l \frac{\Gamma(\beta+l+1)}{\Gamma(\beta-l-1)} \frac{2^{-l}}{l!} x^{\beta-l}. \end{aligned} \tag{C.2}$$

Changing l into $l+1$ in (C.2) we obtain the differential equation

$$x^2 Q''_\beta(x) - 2x(\beta - x) Q'_\beta(x) + 2\beta(1-x) Q_\beta(x) = 0. \tag{C.3}$$

We define the functions,

$$Q_{\beta,k}(x) = (-2)^k x^{\beta-k} \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(\frac{Q_\beta(x)}{x^\beta}\right), \tag{C.4}$$

$$Q_{\beta,0}(x) = Q_\beta(x), \tag{C.5}$$

that is

$$Q_{\beta,k}(x) = \sum_{l=0}^{\infty} (-)^{l+k} \frac{\Gamma(\beta+l+k)}{\Gamma(\beta-l-k)} \frac{2^{-l}}{l!} x^{\beta-l-k}. \tag{C.6}$$

From (C.4) we obtain the recurrence:

$$Q_{\beta,k}(x) = 2(\beta-k+1) Q_{\beta,k-1}(x) - 2x Q'_{\beta,k-1}(x). \tag{C.7}$$

For instance,

$$Q_{\beta,1}(x) = 2\beta Q_{\beta,0}(x) - 2x Q'_{\beta,0}(x) \tag{C.8}$$

and

$$Q_{\beta,2}(x) = 2(\beta-1) Q_{\beta,1}(x) - 2x Q'_{\beta,1}(x), \tag{C.9}$$

$$Q_{\beta,2}(x) = 4x^2 Q''_{\beta,0}(x) - 8x(\beta-1) Q'_{\beta,0}(x) + 4\beta(\beta-1) Q_{\beta,0}(x). \tag{C.10}$$

Thus, we have, from (C.8) and (C.10), the expressions $Q'_{\beta,0}(x)$ and $Q''_{\beta,0}(x)$ in terms of $Q_{\beta,1}(x)$ and $Q_{\beta,2}(x)$. Equation (C.3) becomes

$$Q_{\beta,2}(x) + 4(1-x) Q_{\beta,1}(x) - 4\beta(\beta-1) Q_{\beta,0}(x) = 0. \tag{C.11}$$

From (C.11), by derivatives $\frac{\mathrm{d}}{\mathrm{d}x}$ and recurrence, it is easy to show that

$$Q_{\beta,k+2}(x) + 4(k+1-x) Q_{\beta,k+1}(x) - 4[\beta(\beta-1) - k(k+1)] Q_{\beta,k}(x) = 0. \tag{C.12}$$

C.2. Calogero $n = 3$

We consider the Calogero differential operator,

$$H_{\mathrm{Calogero}} = \sum_{i=1}^{3} \frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + 2\beta \sum_{i<j} \frac{1}{x_i - x_j} \left(\frac{\mathrm{d}}{\mathrm{d}x_i} - \frac{\mathrm{d}}{\mathrm{d}x_j} \right), \tag{C.13}$$

and we look for solutions

$$H_{\mathrm{Calogero}} \Phi\left(x_i, y_i\right) = \left(\sum_i y_i^2 \right) \Phi\left(x_i, y_i\right), \tag{C.14}$$

where the solutions $\Phi\left(x_i, y_i\right)$ have a certain number of symmetry properties described somewhere else. We write

$$\Phi\left(x_i, y_i\right) = f\left(x_i, y_i\right) \mathrm{e}^{\sum_{i=1}^{3} x_i y_i} \tag{C.15}$$

so that equations (C.13) and (C.14) become

$$D = \sum_{i=1}^{3} \frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + 2\beta \sum_{i<j} \frac{1}{x_i - x_j} \left(\frac{\mathrm{d}}{\mathrm{d}x_i} - \frac{\mathrm{d}}{\mathrm{d}x_j} + y_i - y_j \right) + 2 \sum_{i=1}^{3} y_i \frac{\mathrm{d}}{\mathrm{d}x_i} \tag{C.16}$$

and

$$D f\left(x_i, y_i\right) = 0. \tag{C.17}$$

Let us introduce the variables:

$$a = \tfrac{1}{2}\left(x_1 - x_2\right)\left(y_1 - y_2\right) = \left(x_1 - x_2\right) Y_{12} \tag{C.18}$$

$$b = \tfrac{1}{2}\left(x_1 - x_3\right)\left(y_1 - y_3\right) = \left(x_1 - x_3\right) Y_{13} \tag{C.19}$$

$$c = \tfrac{1}{2}\left(x_2 - x_3\right)\left(y_2 - y_3\right) = \left(x_2 - x_3\right) Y_{23,} \tag{C.20}$$

where

$$Y_{ij} = -Y_{ji}, \tag{C.21}$$

and we look for solutions of the type $f\left(a, b, c\right)$. We have

$$\frac{\mathrm{d}f}{\mathrm{d}x_1} = Y_{12} f_a' + Y_{13} f_b' \tag{C.22}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x_2} = Y_{21} f_a' + Y_{23} f_c' \tag{C.23}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x_3} = Y_{31} f_b' + Y_{32} f_c' \tag{C.24}$$

and

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x_1^2} = Y_{12}^2 f_{a^2}'' + 2 Y_{12} Y_{13} f_{ab}'' + Y_{13}^2 f_{b^2}'' \tag{C.25}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x_2^2} = Y_{21}^2 f_{a^2}'' + 2 Y_{21} Y_{23} f_{ac}'' + Y_{23}^2 f_{c^2}'' \tag{C.26}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x_3^2} = Y_{31}^2 f_{b^2}'' + 2 Y_{31} Y_{32} f_{bc}'' + Y_{32}^2 f_{c^2}''. \tag{C.27}$$

Equations (C.16) and (C.17) become

$$Df\,(a,b,c) = D_1 f\,(a,b,c) + D_2 f\,(a,b,c) = 0 \tag{C.28}$$

$$D_1 f\,(a,b,c) = Y_{12}^2\left[2f''_{a^2} + 4(\frac{\beta}{a}+1)\,f'_a + 4\frac{\beta}{a}f\right] + \text{circ.perm.} \tag{C.29}$$

$$D_2 f\,(a,b,c) = 2Y_{12}Y_{13}\left[f''_{ab} + \frac{\beta}{a}f'_b + \frac{\beta}{b}f'_a\right] + \text{circ.perm.} \tag{C.30}$$

We now try the functions,

$$f\,(a,b,c) = \frac{Q_{\beta,k}\,(a)}{a^{2\beta}}\frac{Q_{\beta,k}\,(b)}{b^{2\beta}}\frac{Q_{\beta,k}\,(c)}{c^{2\beta}} = \frac{\{k,k,k\}}{(abc)^{2\beta}}, \tag{C.31}$$

where $Q_{\beta,k}\,(x)$ are defined in (C.4)–(C.6). We consider

$$f\,(a) = \frac{Q_{\beta,k}\,(a)}{a^{2\beta}}, \tag{C.32}$$

and we have

$$2f'_a + 2\frac{\beta}{a}f = \frac{2aQ'_{\beta,k}\,(a) - 2\beta Q_{\beta,k}\,(a)}{a^{2\beta+1}} = -\frac{2kQ_{\beta,k}\,(a) + Q_{\beta,k+1}\,(a)}{a^{2\beta+1}}. \tag{C.33}$$

By derivation we obtain

$$2f''_{a^2} + 2\frac{\beta}{a}f'_a - 2\frac{\beta}{a^2}f = \frac{1}{a^{2\beta+2}}\begin{bmatrix}\frac{1}{2}Q_{\beta,k+2}\,(a) + (\beta+2k+2)\,Q_{\beta,k+1}\,(a)\\ +2k\,(\beta+k+1)\,Q_{\beta,k}\,(a)\end{bmatrix}, \tag{C.34}$$

so that

$$2f''_{a^2} + 4\frac{\beta}{a}f'_a + \frac{2\beta\,(\beta-1)}{a^2}f = \frac{1}{a^{2\beta+2}}\begin{bmatrix}\frac{1}{2}Q_{\beta,k+2}\,(a) + 2\,(k+1)\,Q_{\beta,k+1}\,(a)\\ +2k\,(k+1)\,Q_{\beta,k}\,(a)\end{bmatrix} \tag{C.35}$$

$$2f''_{a^2} + 4\frac{\beta}{a}f'_a = \frac{1}{a^{2\beta+2}}\begin{bmatrix}\frac{1}{2}Q_{\beta,k+2}\,(a) + 2\,(k+1)\,Q_{\beta,k+1}\,(a)\\ -2\,[\beta\,(\beta-1) - k\,(k+1)]\,Q_{\beta,k}\,(a)\end{bmatrix}. \tag{C.36}$$

Finally, we obtain

$$2f''_{a^2} + 4\left(\frac{\beta}{a}+1\right)f'_a + 4\frac{\beta}{a}f = \frac{1}{2a^{2\beta+2}}\begin{bmatrix}Q_{\beta,k+2}\,(a) + 4\,(k+1-a)\,Q_{\beta,k+1}\,(a)\\ -4\,[\beta\,(\beta-1) - k\,(k+1-2a)]\,Q_{\beta,k}\,(a)\end{bmatrix}. \tag{C.37}$$

Now, we use the recurrence relation (C.12) and get the simple result

$$2f''_{a^2} + 4(\frac{\beta}{a}+1)f'_a + 4\frac{\beta}{a}f = -\frac{4k}{a^{2\beta+1}}Q_{\beta,k}(a). \tag{C.38}$$

We just proved that

$$D_1\frac{\{k,k,k\}}{(abc)^{2\beta}} = -4k\left(\frac{Y_{12}^2}{a} + \frac{Y_{31}^2}{b} + \frac{Y_{23}^2}{c}\right)\frac{\{k,k,k\}}{(abc)^{2\beta}}. \tag{C.39}$$

We further transform the result (C.39). We have

$$\frac{Y_{12}^2}{a} = \frac{Y_{12}}{(x_1-x_2)} = \frac{Y_{12}}{\Delta\,(x)}\,(x_1-x_3)\,(x_2-x_3) \tag{C.40}$$

$$\frac{Y_{12}^2}{a} = \frac{Y_{12}}{\Delta\,(x)}[F - (x_1-x_2)^2], \tag{C.41}$$

where

$$\Delta(x) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3), \tag{C.42}$$

$$F = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3. \tag{C.43}$$

By circular permutation we also have

$$\frac{Y_{31}^2}{b} = \frac{Y_{31}}{\Delta(x)}[F - (x_3 - x_1)^2] \tag{C.44}$$

$$\frac{Y_{23}^2}{c} = \frac{Y_{23}}{\Delta(x)}[F - (x_2 - x_3)^2]. \tag{C.45}$$

We note that in (C.39) the quantity F disappears since

$$Y_{12} + Y_{23} + Y_{31} = 0. \tag{C.46}$$

We may write now

$$D_1 \frac{\{k, k, k\}}{(abc)^{2\beta}} = \frac{4k}{\Delta(x)}[(x_1 - x_2)a + \text{circ.perm.}]\frac{\{k, k, k\}}{(abc)^{2\beta}}. \tag{C.47}$$

We now consider

$$D_2 \frac{\{k, k, k\}}{(abc)^{2\beta}}. \tag{C.48}$$

Using

$$f(a, b, c) = f(a) f(b) f(c), \tag{C.49}$$

we have

$$f''_{ab}(a, b, c) + \frac{\beta}{a} f'_b(a, b, c) + \frac{\beta}{b} f'_a(a, b, c) = \left[\begin{matrix} \left(f'_a(a) + \frac{\beta f(a)}{a}\right)\left(f'_b(b) + \frac{\beta f(b)}{b}\right) \\ -\frac{\beta^2 f(a) f(b)}{ab} \end{matrix}\right] f(c) \tag{C.50}$$

but in (C.30) the term $\frac{\beta^2}{ab} f(a) f(b) f(c)$ disappears since

$$\frac{Y_{12}Y_{13}}{ab} + \text{circ.perm.} = \frac{x_2 - x_3}{\Delta(x)} + \text{circ.perm.} = 0. \tag{C.51}$$

Then, from (C.33) we get

$$= \left[\begin{matrix} 2Y_{12}Y_{13} \frac{\left[kQ_{\beta,k}(a) + \frac{1}{2}Q_{\beta,k+1}(a)\right]}{a^{2\beta+1}} \frac{\left[kQ_{\beta,k}(b) + \frac{1}{2}Q_{\beta,k+1}(b)\right]}{b^{2\beta+1}} \frac{Q_{\beta,k}(c)}{c^{2\beta}} \\ +\text{circ.perm.} \end{matrix}\right]. \tag{C.52}$$

Again, the term containing $Q_{\beta,k}(a)\, Q_{\beta,k}(b)\, Q_{\beta,k}(c)$ disappears by (C.51). We write

$$D_2 \frac{\{k, k, k\}}{(abc)^{2\beta}} = \frac{1}{\Delta(x)} \frac{1}{(abc)^{2\beta}} \left[(x_2 - x_3) \left[\begin{matrix} \frac{1}{2}\{k+1, k+1, k\} \\ +k\{k+1, k, k\} + k\{k, k+1, k\} \end{matrix}\right] + \text{circ.perm.} \right]. \tag{C.53}$$

We note that

$$(x_2 - x_3)\left[\{k+1, k, k\} + \{k, k+1, k\} + \{k, k, k+1\}\right] + \text{circ.perm.} = 0, \tag{C.54}$$

so that

$$D_2 \frac{\{k, k, k\}}{(abc)^{2\beta}} = \frac{1}{\Delta(x)} \frac{1}{(abc)^{2\beta}} \left[(x_2 - x_3) \left[\begin{matrix} \frac{1}{2}\{k+1, k+1, k\} \\ -k\{k, k, k+1\} \end{matrix}\right] + \text{circ.perm.} \right]. \tag{C.55}$$

We now collect D_1 and D_2. From (C.28), (C.31), (C.47) and (C.55), we obtain

$$D\frac{\{k,k,k\}}{(abc)^{2\beta}} = \frac{1}{\Delta(x)}\frac{1}{(abc)^{2\beta}}\left[\begin{array}{c}(x_2-x_3)\left[\begin{array}{c}4kc\{k,k,k\}+\frac{1}{2}\{k+1,k+1,k\}\\ -k\{k,k,k+1\}\end{array}\right]\\ +\text{circ.perm}\end{array}\right]. \tag{C.56}$$

Again,

$$(x_2-x_3)\{k,k,k\}+\text{circ.perm}=0 \tag{C.57}$$

so that

$$D\frac{\{k,k,k\}}{(abc)^{2\beta}} = \frac{1}{\Delta(x)}\frac{1}{(abc)^{2\beta}}\left[\begin{array}{c}(x_2-x_3)\left[\begin{array}{c}4k(c-k)\{k,k,k\}\\ +\frac{1}{2}\{k+1,k+1,k\}-k\{k,k,k+1\}\end{array}\right]\\ +\text{circ.perm}\end{array}\right]. \tag{C.58}$$

We now use equation (10) and write

$$D\frac{\{k,k,k\}}{(abc)^{2\beta}} = \frac{1}{\Delta(x)}\frac{1}{(abc)^{2\beta}}\left[\begin{array}{c}(x_2-x_3)\left[\frac{1}{2}\{k+1,k+1,k\}\right.\\ \left.-4k(\beta-k)(\beta+k-1)\{k,k,k-1\}\right]\\ +\text{circ.perm}\end{array}\right]. \tag{C.59}$$

Consequently, we obtain the remarkable result,

$$D\left[\frac{\Gamma(\beta-k)}{\Gamma(\beta+k)}\frac{1}{8^k k!}\frac{\{k,k,k\}}{(abc)^{2\beta}}\right] = \frac{1}{2\Delta(x)}\frac{1}{(abc)^{2\beta}}\left[\begin{array}{c}(x_2-x_3)\left[\begin{array}{c}\frac{\Gamma(\beta-k)}{\Gamma(\beta+k)}\frac{1}{8^k k!}\{k+1,k+1,k\}\\ -\frac{\Gamma(\beta-k+1)}{\Gamma(\beta+k-1)}\frac{1}{8^{k-1}(k-1)!}\{k,k,k-1\}\end{array}\right]\\ +\text{circ.perm}\end{array}\right]. \tag{C.60}$$

Clearly enough, we define for β not the integer

$$f(abc) = \sum_{k=0}^{\infty}\frac{\Gamma(\beta-k)}{\Gamma(\beta+k)}\frac{1}{8^k k!}\frac{\{k,k,k\}}{(abc)^{2\beta}} \tag{C.61}$$

then

$$Df(abc) = 0+\text{circ.perm}. \tag{C.62}$$

Now, if β is an integer

$$Q_{\beta,k\geqslant\beta}(x) = 0 \tag{C.63}$$

and we define

$$f(abc) = \sum_{k=0}^{\beta-1}\frac{\Gamma(\beta-k)}{\Gamma(\beta+k)}\frac{1}{8^k k!}\frac{\{k,k,k\}}{(abc)^{2\beta}}, \tag{C.64}$$

so that

$$Df(abc) = 0+\text{circ.perm}. \tag{C.65}$$

We proved that a solution to (C.14) is

$$\Phi(x_i,y_i) = \left[\sum_{k=0}^{\infty\,\text{or}\,\beta-1}\frac{\Gamma(\beta-k)}{\Gamma(\beta+k)}\frac{1}{8^k k!}\frac{\{k,k,k\}}{(abc)^{2\beta}}\right]e^{\sum_{i=1}^{3}x_i y_i} \tag{C.66}$$

$$H_{\text{Calogero}}\Phi(x_i,y_i) = \left(\sum_i y_i^2\right)\Phi(x_i,y_i), \tag{C.67}$$

where H_{Calogero} is given in (C.13).

References

[1] Abramowitz M and Stegun A 1974 *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables* (New York: Dover)

[2] Anderson G A 1965 An asymptotic expansion for the distribution of the latent roots of the estimated covariance matrix *Ann. Math. Stat.* **36** 1153–73

[3] Awata H, Matsuo Y, Odake S and Shiraishi J 1995 *Phys. Lett.* B **347** 49

[4] Bingham C 1972 An asymptotic expansion for the distribution of the eigenvalues of a 3 by 3 Wishart matrix *Ann. Math. Stat.* **43** 1498–506

[5] Baker T H and Forrester P J 1997 The Calogero–Sutherland model and generalized classical polynomials *Commun. Math. Phys.* **188** 175–216

[6] Brezin E and Hikami S 2002 An extension of the Harish-Chandra–Itzykson–Zuber integral arXiv: math-ph/0208002

[7] Calogero F 1969 Ground state of a one-dimensional N-body system *J. Math. Phys.* **10** 2197–200

[8] Carlitz L 1957 A note on the Bessel polynomials *Duke Math. J.* **24** 151–62

[9] Chekhov L and Eynard B 2006 Matrix eigenvalue model: Feynman graph technique for all genera *J. High Energy Phys.* JHEP12(2006)026 (arXiv:math-ph/0604014)

[10] Desrosiers P 2008 Duality in random matrix ensembles for all Beta arXiv:0801.3438

[11] Duistermaat J J and Heckman G J 1982 *Inv. Math.* **69** 259

[12] Dunkle C F 1989 *Transl. Am. Math. Soc.* **311** 167

[13] Dyson F J 1962 Statistical theory of the energy levels of complex systems: I *J. Math. Phys.* **3** 140–56

[14] Harish-Chandra 1957 *Am. J. Math.* **79** 87–120

[15] Eynard B 2004 A short note about Morozov's formula arXiv:math-ph/0406063

[16] Eynard B 2001 Asymptotics of skew orthogonal polynomials *J. Phys. A: Math. Gen.* **34** 7591 (arXiv:cond-mat/0012046)

[17] Eynard B and Prats Ferrer A 2005 *Commun. Math. Phys.* **264** 115–44 (arXiv:hep-th/0502041) (erratum to be published)

[18] Ginibre J 1965 *J. Math. Phys.* **6** 440
Girko V 1985 *Theor. Prob. Appl.* **29** 694

[19] Herz C 1955 *Ann. Math.* **61** 474–523

[20] Kohler H and Guhr T 2002 *J. Math. Phys.* **43** 2707 (arXiv:math-ph/0011007)

[21] Kohler H and Guhr T 2005 Supersymmetric extensions of Calogero–Moser–Sutherland-like models: construction and some solutions *J. Phys. A: Math. Gen.* **38** 9891–915

[22] Itzykson C and Zuber J-B 1980 *J. Math. Phys.* **21** 411–21

[23] Jevicki A and Sakita B 1980 *Nucl. Phys.* B **165** 511

[24] Jevicki A and Sakita B 1981 *Nucl. Phys.* B **185** 89

[25] Jevicki A 1990 Collective field theory and Schwinger–Dyson equations in matrix models *Proc. Meeting 'Symmetries, Quarks and Strings' (City College, NY, 1–2 Oct.)* (*Preprint* Brown-HET-777)

[26] Knapp A W 1986 *Representation Theory of Semisimple Groups* (Princeton, NJ: Princeton University Press)

[27] Krall H L and Fink O 1948 A new class of orthogonal polynomials: the Bessel polynomials *Trans. Am. Math. Soc.* **65** 100–15

[28] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Clarendon)

[29] Mehta M L 1991 *Random Matrices* 3rd edn (New York: Academic)

[30] Mehta M L 1981 A method of integration over matrix variables *Commun. Math. Phys.* **79** 3

[31] Migdal A A 1983 *Phys. Rep.* **102** 199
David F 1990 *Mod. Phys. Lett.* A **5** 1019

[32] Morozov A 1992 *Mod. Phys. Lett.* A **7** 3503–8 (arXiv:hep-th/9209074)

[33] Prats Ferrer A, Eynard B, Di Francesco P and Zuber J-B 2009 Correlation functions of Harish-Chandra integrals over the orthogonal and the symplectic groups *J. Stat. Phys.* **129** 885–935 (arXiv:math-ph/0610049)

[34] Shatashvili S 1993 *Commun. Math. Phys.* **154** 421–32 (arXiv:hep-th/9209083)

[35] Wiegmann P and Zabrodin A 2006 Large N expansion for the 2D Dyson gas arXiv:hep-th/0601009

[36] Zuber J B 2008 On the large N limit of matrix integrals over the orthogonal group arXiv:0805.0315